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A LINEAR METHOD FOR LOCAL STRUCTURAL MODIFICATION

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ABSTRACT

Structural modification is a technique to modify dynamic characteristics of a structure by changing its mass, stiffness and damping properties. The most important dynamic characteristics of a structure are its natural frequencies since they dictate the vibration resonances. This paper describes a method for structural modification based on solutions for linear simultaneous equations. The method can be used to determine multiple mass and stiffness modifications of an undamped structural system in order to relocate a resonance or resonances. It analyzed the relationship between the spatial property changes and the natural frequency and mode shape changes. The method provides a more generalized solution for mass and stiffness modification than some earlier analytical work reported in the literature. The main advantage of this method is that it does not rely on a complete eigenvalue solution of the original system to provide exact solution. However, some drawbacks also exist that hinders wide application of this method in structural modification. These drawbacks will be discussed. Examples of implementation of this method will be presented in the paper.

1. INTRODUCTION

Structural modification is a technique to study the effects of physical parameter changes of a structural system on its dynamic properties. These physical parameters are related to the system's mass, stiffness, damping properties or a combination of them. There are two opposite approaches for structural modification. The first one is a direct approach. It is to answer the question of what if some mass or stiffness property changes occur. How will that alter the dynamic properties of a system. The solution to this is usually unique and available. The second approach is about having a desired new dynamic property such as a resonance, how and where to change a system's physical properties to accomplish it. This is an inverse problem and the solution can be non-unique or non-existent. Due to practical constraints, often structural modification is allowed only on a limited locations of a structure. This leads to what is referred to as 'local structural modification'. Many structural modification methods have been reported in the literature [1]-[6].

The overwhelming reason to pursue local structural modification is to improve or optimize the dynamic properties of a structure. Designed on specifications, a structure may not always have satisfied dynamic properties. For instance, its natural frequencies may coincide with some ambient vibration frequencies. Thus, the ability to shift the natural frequencies can significantly lessen structural vibration. This paper proposes a method called "Linear Modification Method (LMM)". This method is capable of relocating the natural frequency/frequencies of a mass-spring like dynamic system by local mass and stiffness changes. This method can also be extended to accommodate optimization of the dynamic characteristics of a structural system.

2. *THEORY*

For an n degree-of-freedom (NDOF) dynamic system, the equation of motion subjected to undamped free vibration can be expressed as:

$$([K]-\omega_r^2[M])\{X\}=\{0\} \tag{2-1}$$

Assume a new natural frequency, ω^*, is needed from structural modification using mass and stiffness changes, then, the equation of motion of the modified system can be expressed as:

$$[([K]+\Delta[K])-\omega^{*2}([M]+\Delta[M])]\{Y^*\}=\{0\} \tag{2-2}$$

where, $\Delta[M]$ and $\Delta[K]$ represent the mass and stiffness modification matrices of the original system, and ω^* and Y^* are the natural frequency and the corresponding mode shape of the modified system respectively.

Eq. (2-2) can be recast into:

$$([K]-\omega^{*2}[M])\{Y^*\}-(\omega^{*2}\Delta[M]-\Delta[K])\{Y^*\}=\{0\} \tag{2-3}$$

The receptance of the original system is defined as:

$$[\alpha(\omega^*)]=([K]-\omega^{*2}[M])^{-1} \tag{2-4}$$

then, Eq. (2-3) can be expressed as:

$$\{Y^*\}=[\alpha(\omega^*)](\omega^{*2}\Delta[M]-\Delta[K])\{Y^*\} \tag{2-5}$$

For the sake of simplicity, assume that mass and stiffness modifications only occur among coordinates i, j and k of the NDOF mass-spring system. Figure 1 shows all the possible links between these three coordinates which are the 3 mass blocks and 6 linear springs. Then Eq. (2-5) will become:

$$
\begin{Bmatrix} Y_i^* \\ Y_j^* \\ Y_k^* \end{Bmatrix} = \left[\alpha\left(\omega^*\right)\right]\left\{ \omega^{*2}\begin{bmatrix} \Delta m_i & 0 & 0 \\ 0 & \Delta m_j & 0 \\ 0 & 0 & \Delta m_k \end{bmatrix} - \right.
$$

$$
\left. \begin{bmatrix} \Delta k_{ii} + \Delta k_{ij} + \Delta k_{ik} & -\Delta k_{ij} & -\Delta k_{ik} \\ -\Delta k_{ij} & \Delta k_{jj} + \Delta k_{ij} + \Delta k_{jk} & -\Delta k_{jk} \\ -\Delta k_{ik} & -\Delta k_{jk} & \Delta k_{kk} + \Delta k_{jk} + \Delta k_{ik} \end{bmatrix} \right\} \begin{Bmatrix} Y_i^* \\ Y_j^* \\ Y_k^* \end{Bmatrix} \tag{2-6}
$$

where, the frequency response functions (FRFs) matrix is:

$$
\left[\alpha(\omega^*)\right] = \begin{bmatrix} \alpha_{ii}\left(\omega^*\right) & \alpha_{ij}\left(\omega^*\right) & \alpha_{ik}\left(\omega^*\right) \\ \alpha_{ji}\left(\omega^*\right) & \alpha_{jj}\left(\omega^*\right) & \alpha_{jk}\left(\omega^*\right) \\ \alpha_{ki}\left(\omega^*\right) & \alpha_{kj}\left(\omega^*\right) & \alpha_{kk}\left(\omega^*\right) \end{bmatrix} \tag{2-7}
$$

In Eq. (2-6), the symbols Δm_i, Δm_j, Δm_k represent the mass modifications and the symbols Δk_{ij}, Δk_{jk}, Δk_{ik} represent the stiffness modifications among coordinates i, j and k, respectively. The symbols Δk_{ii}, Δk_{jj} and Δk_{kk} indicate the stiffness modifications of grounded springs.

Through matrix operation on Eq. (2-6), the general LMM equation (LMME) can be derived as:

$$
\begin{Bmatrix} Y_i^* \\ Y_j^* \\ Y_k^* \end{Bmatrix} = \left[\omega^{*2} Y_i^* \begin{Bmatrix} \alpha_{ii}\left(\omega^*\right) \\ \alpha_{ji}\left(\omega^*\right) \\ \alpha_{ki}\left(\omega^*\right) \end{Bmatrix} \quad \omega^{*2} Y_j^* \begin{Bmatrix} \alpha_{ij}\left(\omega^*\right) \\ \alpha_{jj}\left(\omega^*\right) \\ \alpha_{kj}\left(\omega^*\right) \end{Bmatrix} \quad \omega^{*2} Y_k^* \begin{Bmatrix} \alpha_{ik}\left(\omega^*\right) \\ \alpha_{jk}\left(\omega^*\right) \\ \alpha_{kk}\left(\omega^*\right) \end{Bmatrix} \right. \ldots
$$

$$
-Y_i^* \begin{Bmatrix} \alpha_{ii}\left(\omega^*\right) \\ \alpha_{ji}\left(\omega^*\right) \\ \alpha_{ki}\left(\omega^*\right) \end{Bmatrix} \quad -Y_j^* \begin{Bmatrix} \alpha_{ij}\left(\omega^*\right) \\ \alpha_{jj}\left(\omega^*\right) \\ \alpha_{kj}\left(\omega^*\right) \end{Bmatrix} \quad -Y_k^* \begin{Bmatrix} \alpha_{ik}\left(\omega^*\right) \\ \alpha_{jk}\left(\omega^*\right) \\ \alpha_{kk}\left(\omega^*\right) \end{Bmatrix} \quad -\left(Y_i^* - Y_j^*\right) \begin{Bmatrix} \alpha_{ii}\left(\omega^*\right) - \alpha_{ij}\left(\omega^*\right) \\ \alpha_{ji}\left(\omega^*\right) - \alpha_{jj}\left(\omega^*\right) \\ \alpha_{ki}\left(\omega^*\right) - \alpha_{kj}\left(\omega^*\right) \end{Bmatrix} \ldots
$$

$$
\left. -\left(Y_j^* - Y_k^*\right) \begin{Bmatrix} \alpha_{ij}\left(\omega^*\right) - \alpha_{ik}\left(\omega^*\right) \\ \alpha_{jj}\left(\omega^*\right) - \alpha_{jk}\left(\omega^*\right) \\ \alpha_{kj}\left(\omega^*\right) - \alpha_{kk}\left(\omega^*\right) \end{Bmatrix} \quad -\left(Y_i^* - Y_k^*\right) \begin{Bmatrix} \alpha_{ii}\left(\omega^*\right) - \alpha_{ik}\left(\omega^*\right) \\ \alpha_{ji}\left(\omega^*\right) - \alpha_{jk}\left(\omega^*\right) \\ \alpha_{ki}\left(\omega^*\right) - \alpha_{kk}\left(\omega^*\right) \end{Bmatrix} \right] \times
$$

$$
\left\{ \Delta m_i \quad \Delta m_j \quad \Delta m_k \quad \Delta k_{ii} \quad \Delta k_{jj} \quad \Delta k_{kk} \quad \Delta k_{ij} \quad \Delta k_{jk} \quad \Delta k_{ik} \right\}^T \tag{2-8}
$$

For a mass-spring system, the LMME can be used for multiple and simultaneous mass and spring modifications to relocate natural frequency/frequencies. The specific modification equations shown in reference [7] can be derived from this general LMME by deleting the corresponding terms. For example, if mass modification is applied at coordinates i, j and k of an NDOF mass-spring system, let the following terms be zero,

$$
\Delta k_{ii} = \Delta k_{jj} = \Delta k_{kk} = \Delta k_{ij} = \Delta k_{jk} = \Delta k_{ik} = 0 \tag{2-9}
$$

Then the LMME will become the same equation for mass only modification derived in [7]:

$$\omega^{*2}\begin{bmatrix}\alpha_{ii}(\omega^*) & \alpha_{ij}(\omega^*) & \alpha_{ik}(\omega^*)\\ \alpha_{ji}(\omega^*) & \alpha_{jj}(\omega^*) & \alpha_{jk}(\omega^*)\\ \alpha_{ki}(\omega^*) & \alpha_{kj}(\omega^*) & \alpha_{kk}(\omega^*)\end{bmatrix}\begin{bmatrix}Y_i^* & 0 & 0\\ 0 & Y_j^* & 0\\ 0 & 0 & Y_k^*\end{bmatrix}\begin{Bmatrix}\Delta m_i\\ \Delta m_j\\ \Delta m_k\end{Bmatrix}=\begin{Bmatrix}Y_i^*\\ Y_j^*\\ Y_k^*\end{Bmatrix} \tag{2-10}$$

For modifying a single-grounded spring, Δk_{ii}, delete the null modification terms, the LMME will become:

$$Y_i^* = -Y_i^*\{\alpha_{ii}(\omega^*)\}\{\Delta k_{ii}\} \tag{2-11}$$

or,

$$\Delta k_{ii} = -\frac{1}{\alpha_{ii}(\omega^*)} \tag{2-12}$$

Eq. (2-12) is identical to the results in references [1] and [6].

In order to obtain the modification solution through the LMME for a specified natural frequency ω*, the mode shape components Y* have to be pre-defined. The FRF data at the modified coordinates are available usually from experiments[8]). Note that in the LMME, if the number of the modification coordinates is less than that of the modification parameters, the linear equations become underdetermined. To overcome this problem, more than one set of pre-defined values of Y* has to be provided. (the procedure are demonstrated in one of the examples below).

3. *NUMERICAL EXAMPLES*

The validity and feasibility of the LMME are verified in this section by using a 6 DOF mass-spring system in Figure 2. This system was chosen to be simple yet representative in order to clearly illustrate the execution and potential of the method. The model properties of the system are listed in Table 1. In the examples below we assume that the FRF data is available.

Case 1. Mass and stiffness modification at coordinates 1, 2 and 3

In this case, structural modification is applied on mass m_2 and springs k_1 and k_2 to relocate the second natural frequency from 63.913 Hz to 65 Hz. This is done by solving a set of linear equations in Eq. (3-1), which is obtained through deleting the corresponding null modification terms in Eq. (2-8):

$$\left[\omega^{*2}Y_j^*\begin{Bmatrix}\alpha_{ij}(\omega^*)\\ \alpha_{jj}(\omega^*)\\ \alpha_{kj}(\omega^*)\end{Bmatrix} \quad -(Y_i^*-Y_j^*)\begin{Bmatrix}\alpha_{ii}(\omega^*)-\alpha_{ij}(\omega^*)\\ \alpha_{ji}(\omega^*)-\alpha_{jj}(\omega^*)\\ \alpha_{ki}(\omega^*)-\alpha_{kj}(\omega^*)\end{Bmatrix}\right.$$
$$\left. -(Y_j^*-Y_k^*)\begin{Bmatrix}\alpha_{ij}(\omega^*)-\alpha_{ik}(\omega^*)\\ \alpha_{jj}(\omega^*)-\alpha_{jk}(\omega^*)\\ \alpha_{kj}(\omega^*)-\alpha_{kk}(\omega^*)\end{Bmatrix}\right]\begin{Bmatrix}\Delta m_j\\ \Delta k_{ij}\\ \Delta k_{jk}\end{Bmatrix}=\begin{Bmatrix}Y_i^*\\ Y_j^*\\ Y_k^*\end{Bmatrix} \tag{3-1}$$

Note that this is a case of three coordinates and three unknowns. The mass and stiffness modifications can be obtained by pre-defining the Y* values. In this case, there are the corresponding second mode shape elements of the original system. The modification results are listed in Table 2. The LMME provides exact solution for the second natural frequency of the modified system. Furthermore, the corresponding mode shape elements (Bold faced in Table 2) of the modified system are identical to that of the original system.

Case 2. Mass and stiffness modification at coordinates 1 and 2

In this case, structural modification is applied on mass m_1, m_2 and springs k_1, k_2. Since this is a case of two coordinates and four unknowns, with one set of pre-defined values ,Y*, underdetermined linear equation set below can be obtained from the LMME:

$$\begin{bmatrix} \omega^{*2}Y_i^*\begin{Bmatrix}\alpha_{ii}(\omega^*)\\ \alpha_{ji}(\omega^*)\end{Bmatrix} & \omega^{*2}Y_j^*\begin{Bmatrix}\alpha_{ij}(\omega^*)\\ \alpha_{jj}(\omega^*)\end{Bmatrix} & -Y_i^*\begin{Bmatrix}\alpha_{ii}(\omega^*)\\ \alpha_{ji}(\omega^*)\end{Bmatrix} & -(Y_i^*-Y_j^*)\begin{Bmatrix}\alpha_{ii}(\omega^*)-\alpha_{ij}(\omega^*)\\ \alpha_{ji}(\omega^*)-\alpha_{jj}(\omega^*)\end{Bmatrix}\end{bmatrix}\begin{Bmatrix}\Delta m_i\\ \Delta m_j\\ \Delta k_{ii}\\ \Delta k_{ij}\end{Bmatrix}=\begin{Bmatrix}Y_i^*\\ Y_j^*\end{Bmatrix} \tag{3-2}$$

In order to provide exact solution, another set of pre-defined values (in this case, $Y^{\#}$) is used. Then, the combined four linear equations with four unknowns will become:

$$\begin{bmatrix} \omega^{*2}Y_i^*\begin{Bmatrix}\alpha_{ii}(\omega^*)\\ \alpha_{ji}(\omega^*)\end{Bmatrix} & \omega^{*2}Y_j^*\begin{Bmatrix}\alpha_{ij}(\omega^*)\\ \alpha_{jj}(\omega^*)\end{Bmatrix} & -Y_i^*\begin{Bmatrix}\alpha_{ii}(\omega^*)\\ \alpha_{ji}(\omega^*)\end{Bmatrix} & -(Y_i^*-Y_j^*)\begin{Bmatrix}\alpha_{ii}(\omega^*)-\alpha_{ij}(\omega^*)\\ \alpha_{ji}(\omega^*)-\alpha_{jj}(\omega^*)\end{Bmatrix} \\ \omega^{\#2}Y_i^{\#}\begin{Bmatrix}\alpha_{ii}(\omega^{\#})\\ \alpha_{ji}(\omega^{\#})\end{Bmatrix} & \omega^{\#2}Y_j^{\#}\begin{Bmatrix}\alpha_{ij}(\omega^{\#})\\ \alpha_{jj}(\omega^{\#})\end{Bmatrix} & -Y_i^{\#}\begin{Bmatrix}\alpha_{ii}(\omega^{\#})\\ \alpha_{ji}(\omega^{\#})\end{Bmatrix} & -(Y_i^{\#}-Y_j^{\#})\begin{Bmatrix}\alpha_{ii}(\omega^{\#})-\alpha_{ij}(\omega^{\#})\\ \alpha_{ji}(\omega^{\#})-\alpha_{jj}(\omega^{\#})\end{Bmatrix}\end{bmatrix}\begin{Bmatrix}\Delta m_i\\ \Delta m_j\\ \Delta k_{ii}\\ \Delta k_{ij}\end{Bmatrix}=\begin{Bmatrix}Y_i^*\\ Y_j^*\\ Y_i^{\#}\\ Y_j^{\#}\end{Bmatrix} \tag{3-3}$$

Table 3 lists the modification results where the two target frequencies are 64 and 76.5 Hz. The two sets of Y values used in Eq. (3-3) are the first two elements of the second and third modes of the original system respectively. Note that the LMME has again provided exact solution in this combined case.

4. CONCLUDING REMARKS

The aim of the structural modification detailed in this paper is to relocate a resonance or several resonances of an undamped mass-spring system using a method called 'Linear Modification Method'. This analysis can be extended to studying other structural modification objectives such as cancellation of a resonance with an anti-resonance and creation of a resonance-free frequency range. The method works well for a mass-spring type of dynamic system. It has also been applied to structural modification of truss type of structures. However, there are presently some drawbacks for the LMME. For instance, if the number of modification parameters is not equal to the number of the coordinates involved or multiples of them, the LMME becomes either underdetermined or overdetermined. This will lead to inaccurate answers for mass and stiffness modifications. These inaccuracies will in turn lead to significant errors on system's modal properties. As a result, a structural modification objective such as a new natural frequency would not be achieved precisely. In addition, further investigations are needed in order to implement the LMME to engineering structures such as beam or plate structures. For these types of structures, modification may involve solving a set of nonlinear equations.

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NOMENCLATURE

The list of symbols described below represents the standard notation used throughout this paper.

$[M]$:	System mass matrix
$[K]$:	System stiffness matrix
$\Delta[M]$:	Mass modification matrix
$\Delta[K]$:	Stiffness modification matrix
Δm_i:	Mass modification value at ith coordinate
Δk_{ii}	Stiffness modification value between ith and ground
Δk_{ij}:	Stiffness modification value between ith and jth coordinates i and j
$\{Y^*\}$:	Displacement vector of modified system in frequency domain
$[\alpha(\omega^*)]$:	Receptance frequency response function matrix
$\alpha_{ij}(\omega^*)$:	Element of $[\alpha(\omega^*)]$ at the ith row and jth column
ω_r:	Natural frequency of rth mode
ω^*:	Natural frequency of modified system

	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6
Natural Freq. (Hz)	30.046	63.913	75.209	91.235	111.531	119.112
Mass Normalized Mode Shapes	0.0575	-0.0528	0.0895	0.0424	-0.0271	0.2887
	0.1301	-0.0555	0.1861	-0.1026	-0.1612	-0.0938
	0.1496	0.0989	-0.0084	-0.1882	0.1764	0.0351
	0.091	0.2554	0.0359	0.1464	-0.0606	-0.0097
	0.1369	-0.1232	0.0616	0.1752	0.1588	-0.0797
	0.1743	-0.0626	-0.2284	0.0101	-0.1152	0.0124

Table 1 The modal properties of the 6 DOF mass-spring system

	Original system	Modification value	Modified system
M2	10 kg	8.97 kg	18.97 kg
K12	1000 kN/m	2664.839 kN/m	3664.839 kN/m
K23	1000 kN/m	443.583kN/m	1443.583 kN/m

	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6
Natural Freq. (Hz)	31.0971	**65.0000**	70.9788	92.8520	111.1892	151.0338
Mode Shapes	0.0826	**0.0528**	-0.0991	-0.0082	0.0602	-0.3145
	0.1236	**0.0556**	-0.1543	0.0592	0.0866	0.1055
	0.1481	**-0.0991**	-0.0003	0.1808	-0.2383	-0.0279
	0.0916	-0.2984	-0.0345	-0.1289	0.0827	0.004
	0.1489	0.1131	0.0032	-0.2518	-0.1497	0.035
	0.1836	0.0422	0.2566	0.0506	0.1347	-0.001

Table 2 The modification results of case 1.

	Original system	Modification value	Modified system
M1	10 kg	-5.1957 kg	4.8043 kg
M2	10 kg	-0.2396 kg	9.7604 kg
K1	3000 kN/m	-719.4322 kN/m	2280567.7998 kN/m
K2	1000 kN/m	204.1879 kN/m	1204.1879 kN/m

	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6
Natural Freq. (Hz)	29.5871	**64.0000**	**76.5000**	91.9461	112.5307	159.2177
Mode Shapes	0.071	**-0.0528**	**0.0939**	0.0219	0.0279	-0.323
	0.134	**-0.0556**	**0.1953**	-0.0994	0.1832	0.0545
	0.1535	0.1021	-0.0096	-0.204	-0.1782	-0.0073
	0.0928	0.2667	0.0309	0.1525	0.0594	0.0009
	0.1453	-0.1289	0.0818	0.1829	-0.1624	0.0389
	0.1806	-0.0699	-0.2328	0.0158	0.1136	-0.0039

Table 3 The modification results of case 2.

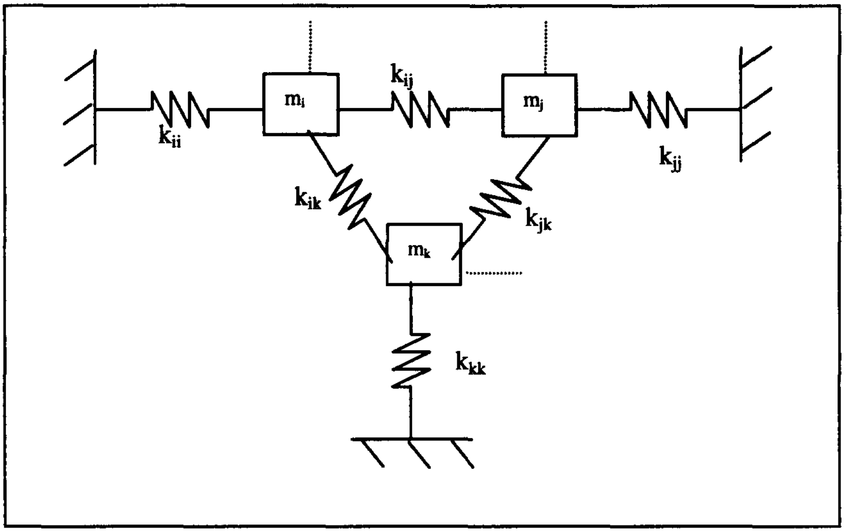

Figure 1. The modification parameters of the NDOF mass-spring system

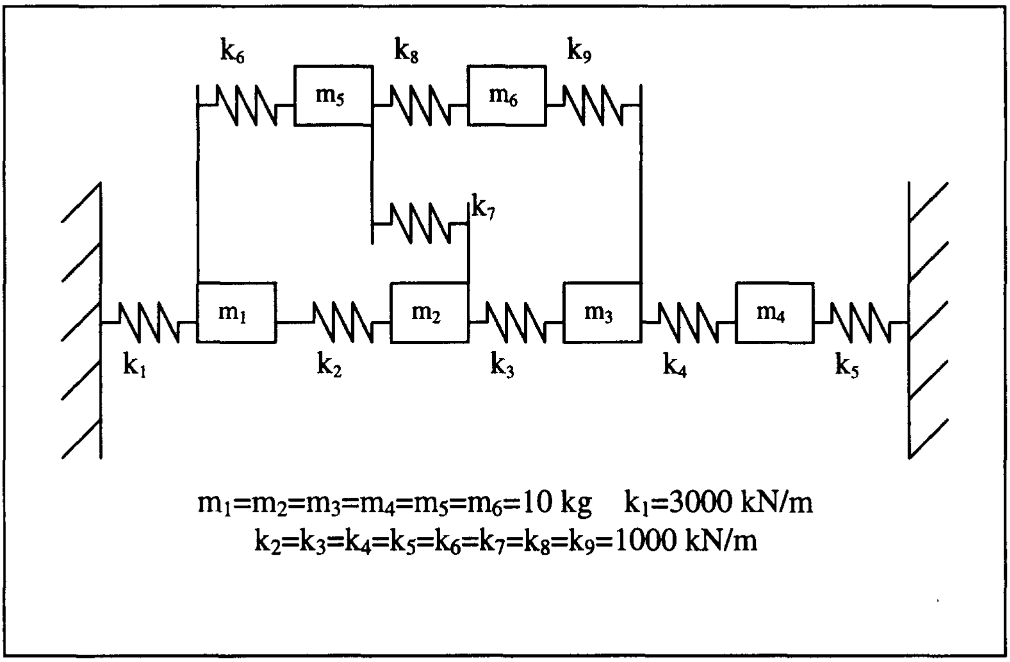

Figure 2. The 6 DOF mass-spring system