A modified frequency domain adaptive filter for active noise control

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ABSTRACT

Frequency domain adaptive filter has potential benefits of fast convergence and low computational load, which make it attractive in the application of active noise control. However, it has been noticed that the commonly used normalized frequency domain block LMS algorithm faces the problem of deterioration of steady-state behavior when the whole control system is noncausal or the adaptive filter is of deficient length. In this paper, an efficient modification of the frequency domain block LMS algorithm is described, and it can be theoretically proven that the proposed algorithm can unconditionally converge to the optimal Wiener solution. Moreover, the convergence behavior is also discussed. The experiment in a short duct active noise control system clearly demonstrate the superiority of the proposed algorithm.

Keywords: Active noise control, Frequency domain adaptive filter, Noncausal condition
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1. INTRODUCTION

The least mean square algorithm (LMS) is the most commonly used adaptive algorithm due to its simplicity and robustness [1-3]. Unfortunately, it suffers from slow convergence rate for reference signal with large eigenvalue disparity, and moreover, its computational burden is heavy in feedforware active noise control systems because the filter length usually has to be set very large.

To overcome the problem of slow convergence, transform domain LMS (TDLMS) algorithms [4-6] have been suggested, which preprocess the reference signal by using orthogonal transforms such as the discrete Fourier transform (DFT), discrete cosine transform (DCT), discrete sine transform (DST), and discrete Hartely transform (DHT), and then set the power normalized step sizes. The improvement of the convergence rate has been proven by many researchers [9-11]. However, the computational burden of the TDLMS algorithms is substantially heavier than that of the LMS algorithm because the orthogonal transforms are often performed for each new input sample. Although the partial updating and the sliding transform techniques can be used to mitigate the problem, the computational burden is still a big challenge for the implementation of the TDLMS algorithms in real time systems.

Apart from the TDLMS algorithms, the DFT can also be used to realize the frequency domain block least mean square (FBLMS) algorithms [1,3], which is a computational efficient implementation of the block LMS (BLMS) algorithm. The computational burden of the FBLMS algorithms are significantly less than that of the LMS algorithm because the fast Fourier transform (FFT) is used to calculate both the block filtering output and the update terms in frequency domain. Furthermore, when the step size of the adaptive filter is normalized by the reference signal power

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in each frequency bin, the convergence speed of the FBLMS algorithms can be significantly increased for reference signal with large power spectra disparity [7,8]. Therefore the bin-normalized FBLMS algorithm is very attractive in active noise control systems.

It has been pointed out that the bin-normalized FBLMS algorithm suffers from the increase of the steady state value of the mean square error in non-causal circumstances [9] or with deficient filter length [10], which are very common in active noise control systems. A frequency-domain Newton's algorithm has been derived in [9] to improve the steady state behavior of the algorithm in non-causal circumstances. However, the extra computational burden for the calculation of the factorization of the step size forms an obstacle to its implementation.

An efficient modification of FBLMS algorithm, named as MFBLMS, has been proposed by the authors [11], which can guarantee an unconditional optimal steady state behavior. In this paper, the convergence property of the modified algorithm will be investigated by using the theory of asymptotical equivalent matrices [12]. Some simulation and experiment results will be presented to prove the efficacy of the proposed algorithm.

2. ANALYSIS OF CONVERGENCE BEHAVIOR OF MFBLMS ALGORITHM

2.1 Description of MFBLMS Algorithm

Let $\mathbf{x}(k) = [x(kN\text{-}N), x(kN\text{-}N+1), \ldots, x(kN+N\text{-}1)]^{\mathrm{T}}$ be the reference signal vector, where the superscript T represents the transpose operation, $\mathbf{w}(k) = [w_0(k), w_1(k), \ldots, w_{N\text{-}1}(k)]^{\mathrm{T}}$ be the N-tap filter, and $\mathbf{d}(k) = [d(kN\text{-}N), d(kN\text{-}N+1), \ldots, d(kN+N\text{-}1)]^{\mathrm{T}}$ be the desired signal vector. Then the error vector in the frequency domain can be described as

$$\mathbf{e}_{\mathrm{f}}(k) = \mathbf{F}\mathbf{G}_{0,N}\mathbf{F}^{-1}\left[\mathbf{d}_{\mathrm{f}}(k) - \mathbf{X}_{\mathrm{f}}(k)\mathbf{w}_{\mathrm{f}}(k)\right] \tag{1}$$

where $\mathbf{F}$ represents a $2N\times2N$ discrete Fourier transform (DFT) matrix, $\mathbf{d}_{\mathrm{f}}(k) = \mathbf{F}[\mathbf{0}_{1\times N}, \mathbf{d}^{\mathrm{T}}(k)]^{\mathrm{T}}$, $\mathbf{X}_{\mathrm{f}}(k) = \mathrm{diag}[\mathbf{x}_{\mathrm{f}}(k)] = \mathrm{diag}[\mathbf{F}\mathbf{x}(k)]$, $\mathbf{w}_{\mathrm{f}}(k) = \mathbf{F}[\mathbf{w}^{\mathrm{T}}(k), \mathbf{0}_{1\times N}]^{\mathrm{T}}$, and

$$\mathbf{G}_{0,N} = \begin{bmatrix} \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} \\ \mathbf{0}_{N\times N} & \mathbf{I}_{N\times N} \end{bmatrix}. \tag{2}$$

The commonly used constrained filter update equation in frequency domain is given by

$$\mathbf{w}_f(k+1) = \mathbf{w}_f(k) + \mathbf{F}\mathbf{G}_{N,0}\mathbf{F}^{-1}\mu\mathbf{M}_f\mathbf{X}_f^{\mathrm{H}}(k)\mathbf{e}_f(k) \tag{3}$$

where the superscript H represents the conjugate transpose operation, μ is a constant step size, $\mathbf{M}_{\mathrm{f}} = \mathrm{diag}[\boldsymbol{\xi}]$ is a diagonal matrix with $\boldsymbol{\xi}$ representing a vector containing the normalizing factors for each frequency bin, and

$$\mathbf{G}_{N,0} = \begin{bmatrix} \mathbf{I}_{N\times N} & \mathbf{0}_{N\times N} \\ \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} \end{bmatrix}. \tag{4}$$

To accelerate the convergence, the normalizing factors can be set as the reciprocal of the reference signal power spectrum as

$$\mathbf{M}_{\mathrm{f}} = \mathrm{diag}\left[\frac{1}{P_0}, \frac{1}{P_1}, \ldots, \frac{1}{P_{2N-1}}\right], \tag{5}$$

where P_i represents the power of the ith frequency bin.

As demonstrated in [9,10], the commonly used NFBLMS algorithm mentioned above faces the problem of deteriorated steady-state performance in noncausal circumstance of when the filter is of deficient length. A simple modification of the algorithm can solve the problem. The updating equation of the proposed MFBLMS algorithm [11] is

$$\mathbf{w}_{\mathrm{f}}(k+1) = \mathbf{w}_{\mathrm{f}}(k) + \mu\mathbf{M}_{\mathrm{f}}\mathbf{F}\mathbf{G}_{N,0}\mathbf{F}^{-1}\mathbf{X}_{\mathrm{f}}^{\mathrm{H}}(k)\mathbf{e}_{\mathrm{f}}(k) \tag{6}$$

Applying inverse Fourier transformation on both sides of (6) leads to

$$\begin{bmatrix} \mathbf{w}(k+1) \\ \mathbf{w}_{\mathrm{nc}}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{w}(k) \\ \mathbf{w}_{\mathrm{nc}}(k) \end{bmatrix} + \mu\mathbf{M}\mathbf{G}_{N,0}\mathbf{X}(k)\begin{bmatrix} \mathbf{0}_{N\times1} \\ \mathbf{e}(k) \end{bmatrix}, \tag{7}$$

where $\mathbf{e}(k) = [e(kN), e(kN+1), \ldots, e(kN+N\text{-}1)]^{\mathrm{T}}$,

$$\mathbf{X}(k)=\mathbf{F}^{-1}\mathbf{X}_{\mathrm{f}}^{\mathrm{H}}(k)\mathbf{F}=\begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2\\ \mathbf{X}_2 & \mathbf{X}_1\end{bmatrix} \tag{8}$$

is a circulant matrix whose first row is $\mathbf{x}(k)$, and

$$\mathbf{M}=\mathbf{F}^{-1}\mathbf{M}_{\mathrm{f}}\mathbf{F}=\begin{bmatrix}\mathbf{M}_1 & \mathbf{M}_2\\ \mathbf{M}_2 & \mathbf{M}_1\end{bmatrix} \tag{9}$$

is also a circulant matrix whose first column is $\mathbf{F}^{-1}\boldsymbol{\xi}$ (the inverse Fourier transform of the normalizing vector), and $\mathrm{w}_{\mathrm{nc}}(k)$ represents the non-causal part of the adaptive filter, which does not influence the filtered output. With simple derivation, it can be found that the updating of the causal part of the filter can be described as

$$\mathbf{w}(k+1)=\mathbf{w}(k)+\mu\mathbf{M}_1\mathbf{X}_2\mathbf{e}(k)\,, \tag{10}$$

then taking expectation on both sides of (10) yields

$$E\left[\mathbf{w}(k+1)\right]=\left[\mathbf{I}_{N\times N}-\mu\mathbf{M}_1\mathbf{R}\right]E\left[\mathbf{w}(k)\right]+\mu\mathbf{M}_1\mathbf{r}\,. \tag{11}$$

The steady state solution of (11) is $E[\mathbf{w}_{\infty}(k)]=\mathbf{R}^{-1}\mathbf{r}$, and obviously, this means an unconditional convergence to the Wiener solution.

2.2 General Convergence Analysis of an ARMA Process

In many applications, the input signal can be modeled by a minimum phase stable autoregressive moving average (ARMA) process, which is generated by passing a unit-variance white noise through a pole-zero transfer function as

$$H(z)=\frac{\sigma^2\prod_{i=0}^{Q}\left(z-q_i\right)}{\prod_{i=0}^{P}\left(z-p_i\right)} \tag{12}$$

where σ^2 represents the variance of the signal, p_i and q_i denote the pole and zero of the transfer function respectively. Note that the modulus of both p_i and q_i are smaller than 1. The autocorrelation sequence of this ARMA process is dominated by damped exponentials (from the real poles) and/or damped sine waves (from the conjugate complex poles), thus it is reasonable to assume that the autocorrelation of the reference signal is a sequence of

$$\mathbf{r}_x=\left\{r_{m,}r_{m-1},\cdots,r_1,r_0,r_1,r_2,\cdots,r_m\right\}, \tag{13}$$

and $r_i = 0$ for $|i| > m$, so that for a sufficiently large $N > m$, the autocorrelation matrix is banded as [12]

$$\mathbf{R}_x=\mathrm{toeplitz}\left\{\left[r_0,r_1,\cdots,r_m,\mathbf{0}_{1\times(N-m-1)}\right]^{\mathrm{T}}\right\} \tag{14}$$

where toeplitz{$\mathbf{r}$} stands for a symmetric Toeplitz matrix having $\mathbf{r}$ as its first column.

The Fourier transform of the autocorrelation sequence $\mathbf{r}_{\mathrm{x}}$ is the power spectrum of the reference signal as

$$P_x\left(\omega\right)=\sum_{k=-m}^{m}r_x(k)e^{-jk\omega} \tag{15}$$

while the reciprocal of $P_x(\omega)$ is the power spectrum of the ARMA process with the transfer function of $1/H(z)$. This inverse ARMA process is also a minimum-phase stable transfer function, and its autocorrelation sequence can also be assumed as

$$\bar{\mathbf{r}}=\left\{\bar{r}_{\bar{m},}\bar{r}_{\bar{m}-1},\cdots,\bar{r}_1,\bar{r}_0,\bar{r}_1,\bar{r}_2,\cdots,\bar{r}_{\bar{m}}\right\}, \tag{16}$$

and $\bar{r}_i=0$ for $|i|>\bar{m}$, so that for a sufficiently large $N>\bar{m}$, the autocorrelation matrix of this ARMA process is also banded as

$$\bar{\mathbf{R}}_x=\mathrm{toeplitz}\left\{\left[\bar{r}_0,\bar{r}_1,\cdots,\bar{r}_{\bar{m}},\mathbf{0}_{1\times(N-\bar{m}-1)}\right]\right\}. \tag{17}$$

When the normalizing factor is set proportional to the reciprocal of $P_x(\omega)$ as shown in(15), from the definition of (9), the matrix $\mathbf{M}$ can be described as

$$\mathbf{M}=\mathrm{circulant}\left\{\left[\bar{r}_0,\bar{r}_1,\cdots,\bar{r}_{\bar{m}},\mathbf{0}_{1\times(2N-2\bar{m})},\bar{r}_{\bar{m}},\bar{r}_{\bar{m}-1},\cdots,\bar{r}_1\right]^{\mathrm{T}}\right\} \tag{18}$$

where circulant{**r**} stands for a circulant matrix having **r** as its first column. From (9), (17) and (18) it can be found out that

$$\mathbf{M}_1 = \bar{\mathbf{R}}_x . \tag{19}$$

It has been proven that the Toeplitz matrix of (14) is asymptotically equivalent to a circulant matrix as [12]

$$\mathbf{C} = \text{circulant}\left\{\left[r_0, r_1, \cdots, r_m, \mathbf{0}_{1\times(N-2m)}, r_m, r_{m-1}, \cdots, r_1\right]^{\mathrm{T}}\right\}. \tag{20}$$

As described in [12], this asymptotic equivalence is abbreviated as $\mathbf{R}_x \sim \mathbf{C}$. Moreover, from the properties of the asymptotic equivalence matrices as shown in [28],

$$\mathbf{R}_x^{-1} \sim \mathbf{C}^{-1} \tag{21}$$

if the norm of $\mathbf{R}_x^{-1}$ and $\mathbf{C}^{-1}$ are both bounded for any N.

The matrix **C** can be decomposed as

$$\mathbf{C} = \mathbf{F}_N^{-1}\mathbf{P}_N\mathbf{F}_N \tag{22}$$

where $\mathbf{F}_N$ represents a $N\times N$ DFT matrix and

$$\mathbf{P}_N = \text{diag}\left\{\mathbf{F}\left[r_0, r_1, \cdots, r_m, \mathbf{0}_{1\times(N-2m)}, r_m, r_{m-1}, \cdots, r_1\right]^{\mathrm{T}}\right\} \tag{23}$$

is a diagonal matrix whose diagonal elements are the power spectra of the reference signal. It is straightforward to find out that

$$\begin{aligned}\mathbf{C}^{-1} &= \mathbf{F}_N^{-1}\mathbf{P}_N^{-1}\mathbf{F}_N \\ &= \text{circulant}\left\{\left[\bar{r}_0, \bar{r}_1, \cdots, \bar{r}_{\bar{m}}, \mathbf{0}_{1\times(N-2\bar{m})}, \bar{r}_{\bar{m}}, \bar{r}_{\bar{m}-1}, \cdots, \bar{r}_1\right]^{\mathrm{T}}\right\}\end{aligned} . \tag{24}$$

Comparing (17) with (24), it can be found out that

$$\mathbf{C}^{-1} \sim \bar{\mathbf{R}}_x . \tag{25}$$

From (19), (21) and (25), it can be found out that

$$\mathbf{M}_1 \sim \mathbf{R}_x^{-1} . \tag{26}$$

Note that $\mathbf{R} = N\mathbf{R}_x$, so that

$$\mathbf{M}_1\mathbf{R} \sim N\mathbf{I}_{N\times N} . \tag{27}$$

Eq. (27) shows clearly that the eigenvalues of the matrix $\mathbf{M}_1\mathbf{R}$ has the tendency to be equally distributed as that of an identity matrix, which should result in a good convergence behavior of the proposed algorithm. However, as pointed out in [12], the result is rather general and does not indicate anything about the convergence of the individual eigenvalues. It is possible that although most of the eigenvalues converge to μN, some high or low eigenvalues still exist. Thus further assumption about the reference signal is needed to obtain stronger results.

3. SIMULATIONS AND EXPERIMENTS

3.1 Simulations

To demonstrate the effectiveness of the proposed method, two simulations are conducted with the same setup as those described in [9] (denoted as Elliott's example hereafter) and [10] (denoted as Wu's example) respectively. For brevity of description, the bin-normalized FBLMS algorithm is abbreviated as NFBLMS, and the proposed bin-normalized method as MFBLMS.

A. Elliott's example

In this example, the reference signal was generated by passing Gaussian white noise with unit variance through a low pass filter with a transfer function $H(z) = [(1-0.5z^{-1})/(1-0.6z^{-1})]^{16}$. The desired signal was one sample ahead of the reference signal, resulting in a typical linear prediction problem. A white noise signal uncorrelated with the reference signal was added to the desired signal so that the maximum attenuation of the mean square error obtained by the Wiener solution is about 18.7 dB. The simulation results were averaged based on 100 independent trials. The adaptive filter length N was 128 and the 256-point FFT were used. For this example, the eigenvalue spread of $\mathbf{M}_1\mathbf{R}$ is 107.6, which is significantly lower than that of $\mathbf{R}_x$, 9808.9. Further exploration of the eigenvalue distribution of $\mathbf{M}_1\mathbf{R}_x$ reveals that 122 eigenvalues are equal to one, which shows the asymptotic equivalence to the identity matrix as proven in Sec. III.A. The step sizes for the NFBLMS algorithm and the MFBLMS algorithm were 0.001 and 0.00005 respectively, both of which were close to the upper limits to guarantee both the fastest convergence speed and stable steady state behaviors. The eigenvalue spread of the NFBLMS algorithm, shown as $\mathbf{R}_{xx}^{\mu N}$ in [7], is 2.3 in this example, so the step size of it can be set substantially larger.

Figure 1 shows the convergence curves of these two algorithms. It is shown clearly that the steady state performance of the NFBLMS algorithm deteriorates seriously with only 2.8 dB attenuation of the least mean square error. The convergence speed of the MFBLMS algorithm is very fast initially with 18 dB attenuation obtained within 1500 samples. After that, the slow mode dominates the convergence process and the optimal steady state performance is achieved at about 20000 samples. The steady state filter coefficients are depicted in Fig. 2. Note that only the first 9 coefficients are depicted since the rest of the coefficients are all very close to 0. The deviation of the NFBLMS algorithm is obvious while the MFBLMS algorithm converges to the Wiener solution.

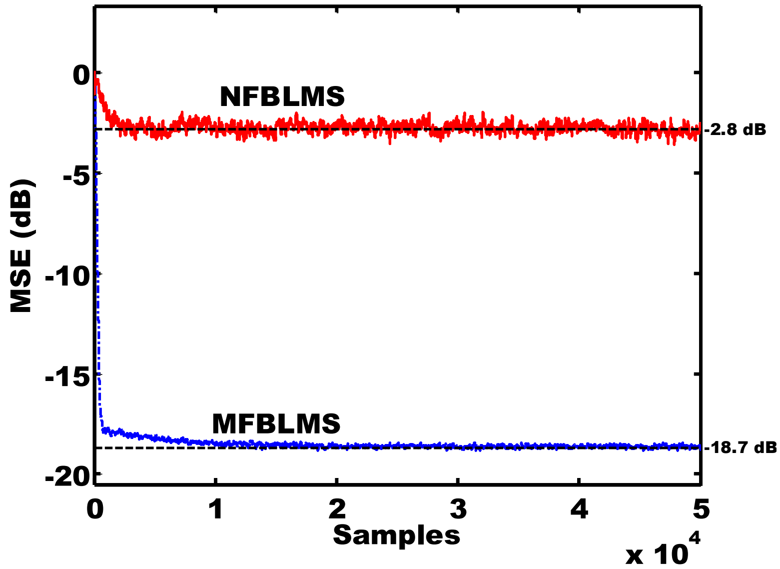

Fig. 1 Convergence of NFBLMS and MFBLMS algorithms for Elliott's example

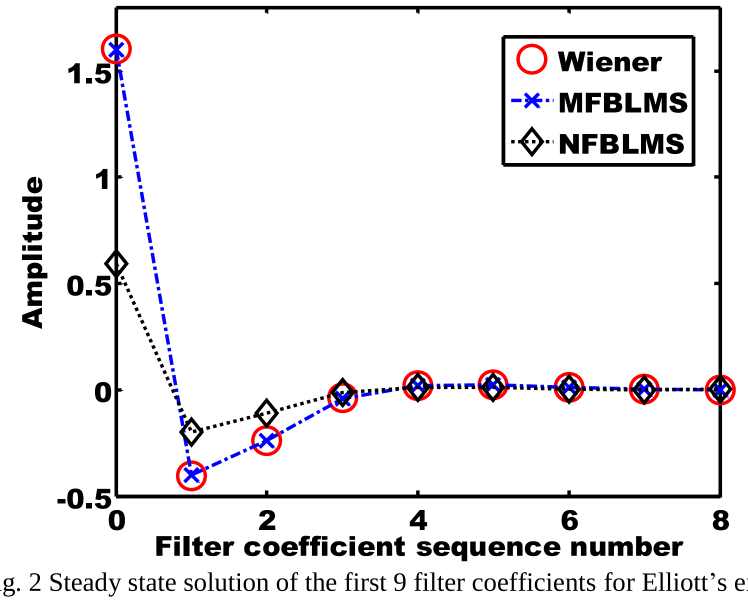

Fig. 2 Steady state solution of the first 9 filter coefficients for Elliott's example

B. Wu's example

In this example, the reference signal was generated by passing Gaussian white noise with unit variance through a 4-tap FIR filter with coefficients [0.1 0.2 -0.4 0.7]. The desired signal was generated by passing the reference signal through a 16-tap FIR filter with coefficients [0.01 0.02 -0.04 -0.08 0.15 -0.3 0.45 0.6 0.6 0.45 -0.3 0.15 -0.08 -0.04 0.02 0.01]. A 10-tap adaptive filter was used, resulting in a typical deficient filter length scenario. The eigenvalue spreads of the MFBLMS algorithm and the NFBLMS algorithm are 1.6 and 1.0 respectively. Both of them are significantly lower than that of $\mathbf{R}_x$, 13.9. The step size of both algorithms was 0.0001, the same as that set in [10]. The simulation results were also averaged by 100 independent trials.

From the convergence curve shown in Fig. 3, it can be found out that the two algorithms have roughly the same convergence speed for the system modeling problem. This is reasonable because the eigenvalue spreads of the two algorithms are very close. However, the MFBLMS algorithm benefits from a lower mean square error. As depicted in Fig. 4, the steady state solution of the NFBLMS algorithm deviates from the Wiener solution especially at the last several points (which is consistent with the result of [10]), while the MFBLMS algorithm converges precisely to the Wiener solution.

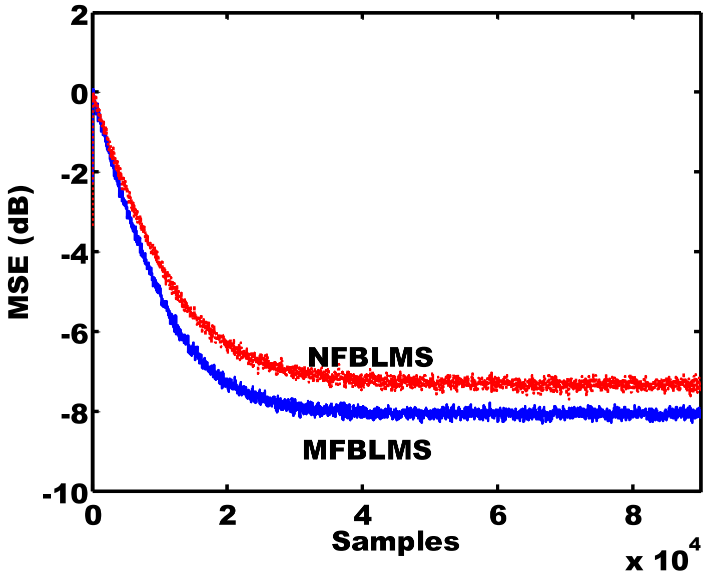

Fig. 3 Convergence of NFBLMS and MFBLMS algorithms for Wu's example

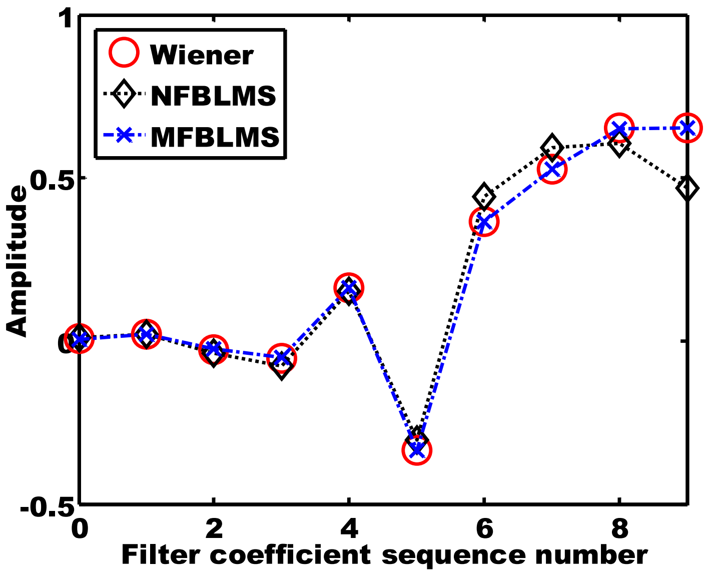

Fig. 4 Steady state solution of the filter coefficients for Wu's example

These simulations clearly demonstrate the superior steady state behavior of the proposed MFBLMS algorithm in both the non-causal and deficient filter length circumstances. Although the convergence speed might be slower than that of the NFBLMS algorithm in some situations, the guaranteed optimal steady state behavior still makes this algorithm a good option for many applications.

3.2 Experiment

A compact duct ANC system is established, with noise source 43 cm away from the error microphone, and the control source 15 cm away from the error microphone. The sampling frequency is 4 kHz. The extra time delay caused by the anti-alias filtering is 5 sampling time intervals; therefore the whole system is destined to be non-causal. The noise used in the experiment is transportation noise captured near a highway. The noise reduction level of different algorithms is shown in Table 1. Note that the FXLMS algorithm is used as a benchmark. It can be seen that the MFBLMS algorithm always performs better than the NFBLMS. More interestingly, with the increase of the filter length, the NFBLMS algorithm even shows the trend of deterioration, which is in accordance with the analysis in [13].

Table 1 Noise reduction level (dB) of different algorithms with different filter length in non-causal condition

N	32	64	128	256
NFBLMS	4.17	5.27	4.09	2.42
MFBLMS	6.09	7.99	8.69	8.76
FXLMS	6.32	8.27	8.66	8.77

4. CONCLUSIONS

In this paper, an efficient modification of FBLMS algorithm, MFBLMS, is introduced, which can guarantee optimal steady-state behavior. This makes the algorithm very attractive in the application of active noise control system. The convergence behavior of the proposed algorithm is analyzed with a general ARMA signal model and the asymptotical equivalent matrices theory. It has been proven that the MFBLMS algorithm tends to have uniform convergence behavior along all the eigen-subspaces. The numerical examples demonstrate the superiority of the MFBLMS algorithm. Furthermore, the experiment in a noncausal short duct active noise control system proves that the proposed algorithm always performs better than the NFBLMS algorithm.

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