Acoustic and flexural wave energy conservation for a thin plate in a fluid

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ABSTRACT

Although the equations of flexural wave motion for a thin plate in a vacuum and a fluid are well known, it is not easy to find a discussion of energy conservation for plate flexural waves, particularly "leaky" waves where a plate and fluid can exchange energy. Nor are formulae easily found for acoustic and flexural wave kinetic energy density, potential energy density and energy density flux including the effect of leaky waves. This paper derives formulae for acoustic and flexural energy densities and energy density fluxes, and finds the energy conservation equation for the coupled thin plate – fluid system

Keywords: Acoustic, Plate, Fluid I-INCE Classification of Subjects Number(s): 21.4, 23, 35.2.2, 42

1. INTRODUCTION

Conservation of energy is sometimes a useful constraint in understanding structural vibration problems. For instance, it's useful to know if radiation from a structure into a fluid can be ignored for near field acoustics because the energy density fluxes within the structure are much larger than the acoustic energy density flux. Even for the simplest system of flexural waves for an infinite thin plate in a fluid, the best known text books do not derive from the wave equation formulae for plate wave energy densities and energy density fluxes (1, 2, 3). This paper fills this basic information gap by deriving formulae for acoustic and flexural wave energy density and energy density flux, and an energy continuity equation for flexural waves coupled to acoustic waves. Regarding the case of a leaky wave whereby a flexural wave coexists with an acoustic wave close to a plate, this paper discusses energy conservation where for example an acoustic leaky wave travels away from a plate but its amplitude decreases exponentially with distance from the plate and increases exponentially with distance along the plate.

2. WAVE ENERGY DENSITY AND ENERGY DENSITY FLUX RELATIONS

Consider an isolated lossless system (i.e. no external applied forces and no energy dissipation) satisfying a wave equation of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial t^2} = 0 \tag{1}$$

where $u(x,t)$ and $v(x,t)$ are complex functions related to the wave amplitude. We seek to identify from eqns. (1) the energy density $U(x,t)$ and energy density flux $\Gamma(x,t)$ that satisfies the conservation of energy equation

$$\frac{\partial \Gamma}{\partial x} + \frac{\partial U}{\partial t} = 0 \tag{2}$$

Equation (2) simply states that the rate of change in energy density U is due to a gradient in the energy density flux Γ. Well known physical concepts allow $U(x,t)$ to be identified as the sum of kinetic energy $U_K(x,t)$ and potential energy $U_P(x,t)$ so that

$$U(x,t) = U_K(x,t) + U_P(x,t) \tag{3}$$

It is tempting to assume that $\Gamma(x,t)$ is a sum of a kinetic energy density flux $\Gamma_K(x,t)$ and a potential energy density flux $\Gamma_P(x,t)$. However identifying formulae for $\Gamma_K(x,t)$ and $\Gamma_P(x,t)$ unambiguously is

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not as straightforward as for $U_K(x,t)$ and $U_P(x,t)$. Instead the derivations below divides $\Gamma(x,t)$ into 2 parts denoted $\Gamma_A(x,t)$ and $\Gamma_B(x,t)$ that are not unambiguously related to $\Gamma_K(x,t)$ and $\Gamma_P(x,t)$, but

$$\Gamma(x,t) = \Gamma_A(x,t) + \Gamma_B(x,t) \tag{4}$$

Energy densities and energy density fluxes are real quantities that are quadratic in $u(x,t)$ and $v(x,t)$. Equation (2) can be constructed from eqn. (1) using only the real parts[2] $u' = \left(u + u^*\right)/2$ 'and $v' = \left(v + v^*\right)/2$. This defines $U(x,t)$ and $\Gamma(x,t)$ applicable at any position x and time t, but often the random phase averaged energy density and energy density flux are also useful.

An energy conservation equation is derived from the real part of eqn. (1),

$$\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 v'}{\partial t^2} = 0 \tag{5}$$

Multiply the LHS of eqn. (5) by $\dot{v}'$, the result extracts the spatial gradient and time derivative terms from the identities,

$$\dot{v}' \frac{\partial^2 u'}{\partial x^2} = \frac{\partial}{\partial x}\left(\dot{v}' \frac{\partial u'}{\partial x}\right) - \left(\frac{\partial \dot{v}'}{\partial x}\frac{\partial u'}{\partial x}\right) \tag{6a}$$

$$\dot{v}' \frac{\partial^2 v'}{\partial t^2} = \frac{1}{2}\frac{\partial}{\partial t}\left(\dot{v}'^2\right) \tag{6b}$$

We then get an energy conservation equation of the form

$$\left(\frac{\partial \Gamma_A}{\partial x} + \frac{\partial U_K}{\partial t}\right) + \left(\frac{\partial \Gamma_B}{\partial x} + \frac{\partial U_P}{\partial t}\right) = 0 \tag{7}$$

where

$$U_K = \frac{1}{2}\dot{v}'^2 \tag{8a}$$

$$\Gamma_A = \dot{v}' \frac{\partial u'}{\partial x} \tag{8b}$$

$$\frac{\partial \Gamma_B}{\partial x} + \frac{\partial U_P}{\partial t} = -\left(\frac{\partial \dot{v}'}{\partial x}\frac{\partial u'}{\partial x}\right) \tag{8c}$$

The generalisations of eqns. (1) to (8) to three dimensions are straightforward by replacing x by spatial vector $\mathbf{r}$, replacing the one dimension spatial derivative by the vector gradient operator ∇ and replacing the second order spatial derivative by the Laplacian operator ∇^2. Also the energy density flux $\mathbf{\Gamma}$ becomes a vector.

The generalised wave equation is

$$\nabla^2 u + \frac{\partial^2 v}{\partial t^2} = 0 \tag{9}$$

The energy conservation eqn. (2) generalises to

$$\nabla.\mathbf{\Gamma} + \frac{\partial U}{\partial t} = 0 \tag{10}$$

Equations (3) and (4) are simply extended to functions of vector $\mathbf{r}$. Eqn. (8a) for the kinetic energy density is unchanged. The formula (8b) for the energy density flux is replaced by

$$\mathbf{\Gamma}_A = \dot{v}' \nabla u' \tag{11}$$

Similarly the three dimensional extension of eqn. (8c) is

$$\nabla.\mathbf{\Gamma}_B + \frac{\partial U_P}{\partial t} = -\nabla \dot{v}'.\nabla u' \tag{12}$$

3. ACOUSTIC WAVES IN A FLUID

3.1 Acoustic energy density and energy density flux relations

The acoustic wave equation is obtained by using in eqn. (9) the relationships

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[2] Any complex quantity q is $q = q' + iq''$ where q' denotes the real part and q'' denotes the imaginary part.

$$u(\mathbf{r},t) = Ap(\mathbf{r},t) \tag{13a}$$

$$v(\mathbf{r},t) = -\frac{1}{c_0^2} A\dot{p}(\mathbf{r},t) \tag{13b}$$

where $p(\mathbf{r},t)$ is the complex acoustic wave pressure, and c_0 is the speed of sound in the bulk fluid (i.e. a large distance from any elastic or other surface). The factor A ensures that U and $\mathbf{\Gamma}$ have units of energy density and energy density flux respectively. For a plane wave of frequency $f = \omega/(2\pi)$ and fluid density ρ_0 , the average energy density flux (Fahy and Giordano Chapter 3, eqn. (3.16) (4)) can be used to find

$$A = \frac{c_0}{\omega\sqrt{\rho_0}} \tag{14}$$

From eqns. (8a), (11) and (13a, b) we find spatial and time dependent quantities

$$U_K = \frac{A^2}{2c_0^4}\left(\frac{1}{2}\left(\dot{p}+\dot{p}^*\right)\right)^2 = \frac{A^2}{4c_0^4}|\dot{p}|^2 + \frac{A^2}{8c_0^4}\left(\dot{p}^2 + \left(\dot{p}^*\right)^2\right) \tag{15a}$$

$$\mathbf{\Gamma}_A = -\frac{A^2}{c_0^2}\left(\frac{1}{2}\left(\dot{p}+\dot{p}^*\right)\nabla\frac{1}{2}\left(p+p^*\right)\right) = -\frac{A^2}{4c_0^2}\left(\dot{p}\nabla p^* + \dot{p}^*\nabla p\right) - \frac{A^2}{4c_0^2}\left(\dot{p}\nabla p + \dot{p}^*\nabla p^*\right) \tag{15b}$$

Equations (12) and (13a, b) lead to

$$\nabla.\mathbf{\Gamma}_B + \frac{\partial U_P}{\partial t} = \frac{A^2}{c_0^2}\frac{1}{2}\nabla\left(\dot{p}+\dot{p}^*\right)\frac{1}{2}\nabla\left(p+p^*\right) = \frac{A^2}{4c_0^2}\frac{\partial\left(|\nabla p|^2\right)}{\partial t} + \frac{A^2}{8c_0^2}\frac{\partial\left((\nabla p)^2 + \left(\nabla p^*\right)^2\right)}{\partial t} \tag{16a}$$

Comparing the LHS and RHS of eqn. (16a) implies that U_P and $\mathbf{\Gamma}_B$ are given by

$$U_P = \frac{A^2}{4c_0^2}\left(|\nabla p|^2 + \frac{1}{2}\left((\nabla p)^2 + \left(\nabla p^*\right)^2\right)\right) \tag{16b}$$

$$\mathbf{\Gamma}_B = \mathbf{0} \tag{16c}$$

Equation (16c) shows that the total acoustic energy density flux is just $\mathbf{\Gamma} = \mathbf{\Gamma}_A$ given by eqn. (15b).

3.2 Plane wave acoustic energy density and energy density flux relations

Consider a single complex pressure wave

$$p(\mathbf{r},t) = p_0 \exp\left[i\mathbf{k_a}.\mathbf{r} - i\omega t + i\phi_a\right] \tag{17}$$

where $\omega = 2\pi f$ for frequency f , $\mathbf{k_a} = \mathbf{k'_a} + i\mathbf{k''_a}$ is the complex acoustic wavenumber vector, ϕ_a [3] is an arbitrary phase, and the pressure amplitude p_0 is real. A wave with a complex wavenumber does not usually occur in isolation but travelling, standing and evanescent waves are needed together to balance energy density and energy density flows. It is possible for $\mathbf{k_a}$ to be complex near a surface such as a plate.

Substituting eqns. (13a, b) and (17) into the wave eqn. (9), for the case where p_0 is position and time independent, we find the constraint on $\mathbf{k_a}$,

$$\mathbf{k_a}.\mathbf{k_a} = k_0^2 \tag{18}$$

where $k_0 = \omega / c_0$. Separate constraints on the real and imaginary parts of eqn. (18) are

$$k_a'^2 - k_a''^2 = k_0^2 \tag{19a}$$

$$\mathbf{k'_a}.\mathbf{k''_a} = 0 \tag{19b}$$

Eqn. (19b) shows that an acoustic wave can have a nonzero imaginary part for the wavenumber (e.g. an evanescent wave) provided the real part and imaginary part of the complex wavenumber vectors are orthogonal. Eqn. (19a) constrains the relative magnitudes of the real and imaginary parts of the wavenumber vector near a surface to conform with the wavenumber in the bulk fluid far from the surface.

The energy density and energy density flux are time dependent with a fluctuation frequency *2f*. The energy density and energy density flux mutually change with time and balance out to satisfy eqn. (10). The details of the derivations are straightforward using eqns. (17), (15a, b) and (16b). For brevity we use a

[3] For a surface wave interacting with the pressure wave in a fluid $\phi_a = \phi_x + \phi_z$ is the sum of the phase ϕ_x of the surface wave and the phase ϕ_z of the pressure wave component travelling perpendicular to the surface.

phase $\psi_a(\mathbf{r},t)$ for acoustic energy density and flux variations defined by

$$\psi_a(\mathbf{r},t) = \psi'_a(\mathbf{r},t) + i\,\psi''_a(\mathbf{r},t) \tag{20a}$$

$$\psi'_a(\mathbf{r},t) = 2(\mathbf{k}'_\mathbf{a}.\mathbf{r} - \omega t) + 2\phi_a,\ \psi''_a(\mathbf{r},t) = 2\mathbf{k}''_\mathbf{a}.\mathbf{r} \tag{20b}$$

The time dependent relationships for the energy densities and energy density fluxes for a single complex wavenumber plane acoustic wave are[4]

$$U_K(\mathbf{r},t) = \frac{A^2}{4c_0^2} k_0^2 p_0^2 \exp[-2\mathbf{k}''_\mathbf{a}.\mathbf{r}](1-\cos(\psi'_a(\mathbf{r},t))) \tag{21a}$$

$$U_P(\mathbf{r},t) = \frac{A^2}{4c_0^2} p_0^2 \exp[-2\mathbf{k}''_\mathbf{a}.\mathbf{r}]\left(\left(k_a'^2 + k_a''^2 - k_0^2\right) + k_0^2(1-\cos(\psi'_a(\mathbf{r},t)))\right) \tag{21b}$$

$$U_S(\mathbf{r}) = \frac{A^2}{2c_0^2} k_a''^2 p_0^2 \exp[-2\mathbf{k}''_\mathbf{a}.\mathbf{r}] \tag{21c}$$

$$\mathbf{\Gamma}(\mathbf{r},t) = \frac{A^2}{2c_0} k_0 p_0^2 \exp[-2\mathbf{k}''_\mathbf{a}.\mathbf{r}](\mathbf{k}'_\mathbf{a}(1-\cos(\psi'_a(\mathbf{r},t))) + \mathbf{k}''_\mathbf{a}\sin(\psi'_a(\mathbf{r},t))) \tag{21d}$$

$k''_a \neq 0$ implies a surface acoustic source (or sink) to be present making the potential energy density larger than the kinetic energy density by the amount $U_S(\mathbf{r})$ (i.e. $U_P = U_K + U_S$). From (21d) this surface source/sink also gives $\mathbf{\Gamma}(\mathbf{r},t)$ a wavenumber $\mathbf{k}''_\mathbf{a}$ perpendicular to $\mathbf{k}'_\mathbf{a}$ but oscillates in sign, averages to zero and hence makes no net contribution to the average energy density flux. All energy densities vary exponentially with distance both parallel to and perpendicular to a surface, so if we require $k''_a > 0$ so that the acoustic energy density decreases with distance from the surface, then for $k'_a > 0$ of a wave travelling away from the surface the energy density increases along the surface in the direction that a wavefront propagates, and for $k'_a < 0$ of a wave travelling towards the surface the energy density decreases along the surface in the direction that a wavefront propagates.

4. FLEXURAL WAVES FOR AN INFINITE THIN FLAT PLATE

4.1 Flexural wave energy density and energy density flux relations

The wave equation for thin plate flexural waves (Fahy and Gardonio Chapter 1, pp. 26 – 27 (4)) is obtained with eqn. (1) by the relations

$$u(x,t) = BD\frac{\partial^2 \xi(x,t)}{\partial x^2} \tag{22a}$$

$$v(x,t) = BM\xi(x,t) \tag{22b}$$

where $\xi(x,t)$ is the complex perpendicular deviation of the plate from the average plane *z=0* at position *x* and time *t*, *D* is the plate bending stiffness per unit area, *M* is the mass per unit area of the plate, and *B* is the factor $B = 1/\sqrt{M}$ that gives correct energy density units.

The expression (8a) for U_K in terms of $\dot{\xi}(x,t)$ using eqn. (22b) leads to

$$U_K(x,t) = \frac{M}{4}\left|\dot{\xi}(x,t)\right|^2 + \frac{M}{8}\left(\dot{\xi}^2(x,t) + \dot{\xi}^{*2}(x,t)\right) \tag{23a}$$

From eqn. (8b) for Γ_A in terms of $\xi(x,t)$ and $\dot{\xi}(x,t)$ using eqns. (22a, b) leads to

$$\Gamma_A = \frac{D}{4}\left(\dot{\xi}^*(x,t)\frac{\partial^3\xi(x,t)}{\partial x^3} + \dot{\xi}(x,t)\frac{\partial^3\xi^*(x,t)}{\partial x^3}\right) + \frac{D}{4}\left(\dot{\xi}(x,t)\frac{\partial^3\xi(x,t)}{\partial x^3} + \dot{\xi}^*(x,t)\frac{\partial^3\xi^*(x,t)}{\partial x^3}\right) \tag{23b}$$

To facilitate identifying the potential energy density and energy density flux Γ_B from eqn. (8c) we use the identity

$$\frac{\partial \dot{v}'}{\partial x}\frac{\partial u'}{\partial x} = \frac{\partial}{\partial x}\left(u'\frac{\partial \dot{v}'}{\partial x}\right) - u'\frac{\partial^2 \dot{v}'}{\partial x^2} \tag{24}$$

Then eqn. (8c) is satisfied by assuming

[4] Since $\mathbf{\Gamma} = \mathbf{\Gamma}_A$ the subscript is dropped.

$$\frac{\partial U_P}{\partial t} = u' \frac{\partial^2 \dot{v}'}{\partial x^2} \tag{25a}$$

$$\Gamma_B = -u' \frac{\partial \dot{v}'}{\partial x} \tag{25b}$$

From eqns. (22a, b) and (25a) we find

$$\frac{\partial U_P}{\partial t} = \frac{D}{4} \frac{\partial \left| \frac{\partial^2 \xi}{\partial x^2} \right|^2}{\partial t} + \frac{D}{8} \frac{\partial}{\partial t} \left(\left(\frac{\partial^2 \xi}{\partial x^2} \right)^2 + \left(\frac{\partial^2 \xi^*}{\partial x^2} \right)^2 \right) \tag{26a}$$

Hence we have

$$U_P = \frac{D}{4} \left| \frac{\partial^2 \xi}{\partial x^2} \right|^2 + \frac{D}{8} \left(\left(\frac{\partial^2 \xi}{\partial x^2} \right)^2 + \left(\frac{\partial^2 \xi^*}{\partial x^2} \right)^2 \right) \tag{26b}$$

Eqn. (26b) is consistent with the static version of the potential energy density stored in a bent thin plate derived in text books such as that in Landau and Lifshitz, eqn. (11.6), p. 46 (5). From eqns. (22a, b) and (25b) we find

$$\Gamma_B = -\frac{D}{4} \left(\frac{\partial \dot{\xi}^*}{\partial x} \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial \dot{\xi}}{\partial x} \frac{\partial^2 \xi^*}{\partial x^2} \right) - \frac{D}{4} \left(\frac{\partial \dot{\xi}}{\partial x} \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial \dot{\xi}^*}{\partial x} \frac{\partial^2 \xi^*}{\partial x^2} \right) \tag{26c}$$

4.2 Plane flexural wave energy density and energy density flux relations

Consider for an infinite flat plate a single complex flexural wave propagating along the x-axis with displacement in the $+z$ directions

$$\xi(x,t) = \xi_0 \exp[ik_x x - i\omega t + i\phi_x] = \xi_0 \exp[-k_x'' x] \exp[ik_x' x - i\omega t + i\phi_x] \tag{27}$$

where $\omega = 2\pi f$ for frequency f , $k_x = k_x' + ik_x''$ is the complex flexural wavenumber, ϕ_x is an arbitrary phase, and the wave amplitude ξ_0 is real and constant. Note that a single wave with a complex wavenumber does not usually occur in isolation owing to the need to conserve energy[5], but the extension of this theory to coherent superpositions of multiple flexural waves is straightforward.

Substituting eqn. (22a, b) and (27) into the wave eqn. (1), we find $k_x^4 = k_f^4$ where $k_f^4 = M\omega^2 / D$ which when separated into real and imaginary parts gives

$$\left(k_x'^2 - k_x''^2\right)^2 - 4k_x'^2 k_x''^2 = k_f^4 \tag{28a}$$

$$k_x' k_x'' \left(k_x'^2 - k_x''^2\right) = 0 \tag{28b}$$

The two wave types satisfying (28a, b) are

$$k_x' = \pm k_f, k_x'' = 0 \tag{29a}$$

$$k_x' = 0, k_x'' = \pm k_f \tag{29b}$$

Wave type eqn. (29a) is the unattenuated flexural wave for a plate in a vacuum, and eqn. (29b) is the corresponding evanescent flexural wave.

Similar to eqns. (20a, b) we use flexural wave energy density and flux phases $\psi'(x,t) = 2(k_x' x - \omega t) + 2\phi_x$ and $\psi''(x,t) = 2k_x'' x$.

Substituting eqn. (27) into the energy density and energy density flux formulae (23a, b) and (26b, c) gives

$$U_K(x,t) = \frac{M\omega^2}{4} \xi_0^2 \exp[-2k_x'' x] \left(1 - \cos(\psi'(x,t))\right) \tag{30a}$$

$$\Gamma_A(x,t) = \frac{D\omega}{2} \xi_0^2 \exp[-2k_x'' x] \left(k_x'' \left(3k_x'^2 - k_x''^2\right) \sin(\psi'(x,t)) + k_x' \left(k_x'^2 - 3k_x''^2\right) \left(1 - \cos(\psi'(x,t))\right)\right) \tag{30b}$$

[5] For instance, a harmonic point or line force applied to an infinite plate creates both a travelling wave with a real wavenumber and an evanescent wave with an imaginary wavenumber. These two waves are correlated by their common driving force.

$$U_P(x,t)=\frac{D}{4}\xi_0^2\exp\left[-2k''_x x\right]\begin{pmatrix}\left(k'^2_x+k''^2_x\right)^2+\left(\left(k'^2_x-k''^2_x\right)^2-4k'^2_x k''^2_x\right)\cos(\psi'(x,t))\\-4k'_x k''_x\left(k'^2_x-k''^2_x\right)\sin(\psi'(x,t))\end{pmatrix}\quad (30c)$$

$$\Gamma_B(x,t)=\frac{D\omega}{2}\xi_0^2\exp\left[-2k''_x x\right]\left(k'_x\left(k'^2_x+k''^2_x\right)+k'_x\left(k'^2_x-3k''^2_x\right)\cos(\psi'(x,t))-k''_x\left(3k'^2_x-k''^2_x\right)\sin(\psi'(x,t))\right)\quad (30d)$$

The total energy density $U=U_K+U_P$ and total energy density flux $\Gamma=\Gamma_A+\Gamma_B$ from eqns. (30a, b, c, d), noting that $M\omega^2=Dk_f^4$, are given by

$$U(x,t)=\frac{D}{4}\xi_0^2\exp\left[-2k''_x x\right]\begin{pmatrix}\left(k_f^4+\left(k'^2_x+k''^2_x\right)^2\right)+\left(\left(\left(k'^2_x-k''^2_x\right)^2-4k'^2_x k''^2_x\right)-k_f^4\right)\cos(\psi'(x,t))\\-4k'_x k''_x\left(k'^2_x-k''^2_x\right)\sin(\psi'(x,t))\end{pmatrix}\quad (31a)$$

$$\Gamma(x,t)=D\omega k'_x\left(k'^2_x-k''^2_x\right)\xi_0^2\exp\left[-2k''_x x\right]\quad (31b)$$

Taking the time derivative of $U(x,t)$ and spatial derivative of $\Gamma(x,t)$, then substituting them into the LHS of eqn. (2) we find,

$$\frac{\partial U(x,t)}{\partial t}+\frac{\partial\Gamma(x,t)}{\partial x}=$$
$$\frac{\omega D}{2}\xi_0^2\exp\left[-2k''_x x\right]\left[\left(\left(\left(k'^2_x-k''^2_x\right)^2-4k'^2_x k''^2_x\right)-k_f^4\right)\sin(\psi'(x,t))+4k'_x k''_x\left(k'^2_x-k''^2_x\right)\left(1-\cos(\psi'(x,t))\right)\right]\quad (32)$$

For a free, unforced plate eqns. (28a, b) apply and then clearly the RHS of (32) is zero. Also from eqns. (28a, b) the total energy density eqn. (31a) and energy density flux eqn. (31b) are time independent for the unforced plate. Interestingly eqn. (31b) shows the energy density flux is always time independent for a single plane flexural wave satisfying eqn. (27). For the evanescent wave $k'_x=0, k''_x\neq 0$ by eqns. (28b) and (31b) we find $\Gamma(x,t)=0$ although $\Gamma_A(x,t)$ and $\Gamma_B(x,t)$ are nonzero but cancel each other out. It is also interesting from eqn. (31b) that $\Gamma(x,t)$ can be negative even if k'_x is positive (i.e. the net energy density flux can be in the opposite direction to the wave velocity).

5. FLEXURAL WAVES FOR AN INFINITE THIN FLAT PLATE IN A FLUID

5.1 Characteristic equation for plate – fluid dispersion relations

The equation of motion for a thin plate with one side in a nonviscous fluid and the other in a vacuum is derived by adding a time and space dependent acoustic pressure generated by the plate to the LHS of eqn. (1) after substituting eqns. (22a, b). This acoustic pressure $p(x,z,t), z\geq 0$ at the plate surface $z=0$ is assumed to be proportional to plate acceleration $\ddot{\xi}(x,t)$ in the +*z* direction after defining a complex fluid mass density $M_a=M'_a+iM''_a$. Hence we have

$$p(x,0,t)=M_a\ddot{\xi}(x,t)\quad (33)$$

For the acoustic wave generated by flexural waves eqn. (17) is now replaced by

$$p(x,z,t)=p_f\exp\left[ik_x x+ik_z z-i\omega t+i\phi_f\right]\quad (34)$$

The plate's acceleration creates a pressure gradient in the fluid so for an acoustically hard plate surface[6]

$$\rho_0\ddot{\xi}(x,t)=-\left(\frac{\partial p(x,z,t)}{\partial z}\right)_{z=0}=-ik_z p(x,0,t)\quad (35)$$

and hence from eqns. (33) and (35)

$$M_a=\frac{i\rho_0}{k_z}=\frac{i\rho_0}{k'^2_z+k''^2_z}\left(k'_z-ik''_z\right)\quad (36a)$$

$$M'_a=\frac{\rho_0 k''_z}{k'^2_z+k''^2_z}\quad (36b)$$

$$M''_a=\frac{\rho_0 k'_z}{k'^2_z+k''^2_z}\quad (36c)$$

[6] This omits compression and shear waves in the plate material that couple to the fluid acoustic and plate flexural waves.

The plate displacement $\xi(x,t)$ is still given by eqn. (27).

Combining eqns. (27), (33) and (34) the amplitude of the pressure wave is given by

$$p_f = |M_a|\omega^2 \xi_0 = \frac{\rho_0 \omega^2}{|k_z|}\xi_0 \tag{37a}$$

and the phase difference between acoustic and flexural waves is

$$\phi_f - \phi_x = -\frac{\pi}{2} \tag{37b}$$

The characteristic equations for a thin plate in a fluid are derived by adding the fluid contribution (often called fluid loading) to the plate mass density and so extending eqns. (28a, b) to

$$\left(k_x'^2 - k_x''^2\right)^2 - 4k_x'^2 k_x''^2 = k_f^4\left(1 + \frac{M_a'}{M}\right) \tag{38a}$$

$$4k_x' k_x''\left(k_x'^2 - k_x''^2\right) = k_f^4 \frac{M_a''}{M} \tag{38b}$$

The x and z components of the complex acoustic wavenumber are constrained by eqns. (19a, b) and lead to

$$k_x'^2 + k_z'^2 = k_0^2 + k_x''^2 + k_z''^2 \tag{39a}$$

$$k_x' k_x'' + k_z' k_z'' = 0 \tag{39b}$$

Eqns. (39a, b) allow k_z' and k_z'' to be related to k_x' and k_x'', hence be eliminated from eqns. (38a, b) that can then be solved.

Whereas for a thin plate in a vacuum there are only the two wave mode dispersion relations (29a, b), eqns. (38a, b) can be rearranged to a fifth order polynomial equation in $k_x'^2$ suggesting there are five different plate – fluid modes with different dispersion relations (Crighton (7)). One well known solution, the Stoneley-Scholte wave, is an unattenuated subsonic wave with a pressure decreasing exponentially with distance from the plate. Other plate-fluid modes are “leaky” waves with complex wavenumbers implying that an infinite plate-fluid system is closed and does not have natural modes corresponding to acoustic far field plane waves moving away from or towards the plate. Some plate-fluid modes are “leaky” below the coincidence frequency[7]. All five plate – fluid modes contribute to vibration excited by a harmonic point or line force (Feit and Lui (8) and Chapman and Sorokin (9)), which then leads to radiation from the region near the excitation area.

5.2 Plate – fluid energy density and energy density flux conservation equation

Conservation of energy equations for a closed plate – fluid system are derived by extensions to equations derived in Section 4. Fluid loading extends eqn. (7) to

$$\left(\frac{\partial \Gamma_A}{\partial x} + \frac{\partial U_K}{\partial t}\right) + \left(\frac{\partial \Gamma_B}{\partial x} + \frac{\partial U_P}{\partial t}\right) + \left(W_F + \frac{\partial U_F}{\partial t}\right) = 0 \tag{40}$$

where U_F is an acoustic energy density in the fluid, related below to M_a', and W_F is the plate – fluid acoustic energy density transfer rate related below to M_a''. Eqn. (40) is derived starting with the fluid loaded extensions to eqns. (1) and (5)

$$\frac{\partial^2 u'}{\partial x^2} + \left(1+\alpha'\right)\frac{\partial^2 v'}{\partial t^2} - \alpha'' \frac{\partial^2 v''}{\partial t^2} = 0 \tag{41}$$

where (22a, b) still identify $u(x,t)$ and $v(x,t)$ in term of $\xi(x,t)$ and

$$\alpha' = \frac{M_a'}{M}, \alpha'' = \frac{M_a''}{M} \tag{42}$$

Multiplying the LHS of eqn. (41) by $\dot{v}'$ and following the same procedure leading to eqn. (7) we find eqn. (40) with the extra terms identified as

$$U_F(x,t) = \frac{M_a'}{2}\dot{\xi}'^2(x,t) = \frac{M_a'}{2}\left(\frac{\dot{\xi} + \dot{\xi}^*}{2}\right)^2 \tag{43a}$$

$$W_F(x,t) = -M_a''\dot{\xi}'(x,t)\ddot{\xi}''(x,t) = -M_a''\left(\frac{\dot{\xi} + \dot{\xi}^*}{2}\right)\left(\frac{\ddot{\xi} - \ddot{\xi}^*}{2i}\right) \tag{43b}$$

Then substituting the plane flexural wave eqn. (27) into eqn. (43a, b) we obtain

[7] The frequency where the flexural wave phase speed equals the speed of sound in the fluid.

$$U_F(x,t) = \frac{M_a'\omega^2}{2}\xi_0^2 \exp[-2k_x''x](1-\cos(\psi'(x,t))) \quad (44a)$$

$$W_F(x,t) = \frac{M_a''\omega^3}{2}\xi_0^2 \exp[-2k_x''x](1-\cos(\psi'(x,t))) \quad (44b)$$

Using eqn. (44a, b) in (40) we obtain the plate – fluid extension to eqn. (32)

$$\frac{\partial U(x,t)}{\partial t} + \frac{\partial \Gamma(x,t)}{\partial x} + \left(W_F + \frac{\partial U_F}{\partial t}\right) = W_{PF}(x,t) \quad (45a)$$

$$W_{PF}(x,t) = \frac{D\omega}{2}\xi_0^2 \exp[-2k_x''x]$$
$$\left[\left(\left(\left(k_x'^2 - k_x''^2\right)^2 - 4k_x'^2k_x''^2\right) - (1+\alpha')k_f^4\right)\sin(\psi'(x,t)) - \left(4k_x'k_x''\left(k_x'^2 - k_x''^2\right) - \alpha''k_f^4\right)(1-\cos(\psi'(x,t)))\right] \quad (45b)$$

When the characteristic eqns. (38a, b) are satisfied, we have $W_{PF}(x,t) = 0$ and recover eqn. (40) for a closed plate-fluid system conserving energy. If $W_{PF}(x,t) \neq 0$, k_x', k_x'', k_z', k_z'' are inconsistent with a closed plate-fluid system and $W_{PF}(x,t) \neq 0$ needs to be cancelled by an external energy source to maintain overall energy conservation. Adding an external energy contribution $-W_{EF}(x,t)$ to the energy density transfer rate on flexural waves on both sides of (45a) to cancel $W_{PF}(x,t)$ gives

$$\frac{\partial U(x,t)}{\partial t} + \frac{\partial \Gamma(x,t)}{\partial x} + \left(W_F(x,t) + \frac{\partial U_F(x,t)}{\partial t}\right) - W_{EF}(x,t) = W_{PF}(x,t) - W_{EF}(x,t) = 0 \quad (46)$$

An example of forced plate – fluid vibration is acoustic absorption and scattering by a plate in a fluid. The application of eqns. (45b) and (46) to this problem is in another paper by the author (6).

6. SUMMARY

Starting with a linear wave equation eqn. (1) able to represent both fluid and thin plate wave media, Sections 2 deduced general formulae for energy density and energy density flux from their conformity with the energy conservation eqn. (2). Whereas fundamental mechanics principles readily allow the kinetic and potential energy densities U_K and U_P respectively to be unambiguously identified, two energy density fluxes Γ_A and Γ_B are needed to describe the propagation of kinetic and potential energy.

Sections 3 applied the energy density and energy density flux formulae to acoustic fields but taking into account the near field effect of a surface resulting in a complex wavenumber vector. A complex wavenumber introduces exponential changes in wave amplitude with distance parallel and perpendicular to the plane of the surface. The acoustic wave equation requires the real and imaginary parts of the wavenumber vector to be orthogonal to each other. In the case of a single plane wave with a complex wavenumber, expressions for the kinetic, potential and surface energy densities are derived. The energy density flux vector has components parallel to and perpendicular to the wavefront corresponding to the imaginary and real parts of the wavenumber vector respectively.

Section 4 derives formulae for the energy density and energy density flux for flexural waves of an infinite thin, flat plate in a vacuum. The net energy density flux for a single flexural plane wave is time independent in contrast to the flux of an acoustic wave which has a time dependent part with frequency $2f$. Also unlike an acoustic wave, a complex wavenumber flexural wave can have an energy density flux in the opposite direction to the wave propagating direction defined by the real part of the wavenumber.

Section 5 combines the theory of Sections 3 and 4 to derive wave dispersion and energy conservation relations for an infinite thin flat plate – fluid system coupling acoustic and flexural waves. Whereas a plate in a vacuum has only two natural wave modes (propagating and evanescent waves), a plate in a fluid has five natural wave modes. The energy conservation equation for the plate-fluid system is extended in Section 5.2 to include external forces such as acoustic excitation by a plane wave treated in another paper by the author (6).

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