A review of the coupling parameter of the Burton and Miller boundary element method

Steffen Marburg[1]

[1] LRT4 Institute of Mechanics, UniBw Munich, 85579 Neubiberg, Germany.

ABSTRACT

The phenomenon of irregular frequencies or spurious modes in boundary element techniques for acoustic problems has been well investigated and a number of methods have been presented to overcome this problem. The most popular methods seem to be the Combined Helmholtz Integral Equation Formulation (CHIEF) and the Burton and Miller formulation. This contribution deals with the latter and its only parameter, i.e. the so-called coupling parameter. It is known for more than four decades that this parameter must be complex for radiation from rigid bodies with particle velocity prescribed. During the 1980ies, a number of authors proposed the value of i/k. This value is proving to be an excellent choice for the high frequency limit. However, it turned out that the optimal choice of the coupling parameter depends on a few more parameters including the harmonic time dependence and the formulation of the integral equation itself. This paper discusses these influences for a specific EAA benchmark case which has only recently been proposed, i.e. the radiatterer.

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1. INTRODUCTION

The Burton and Miller formulation [1] for exterior acoustic problems is well-known since it is free of fictitious resonances, see also [2]. The only parameter of the Burton and Miller formulation is known as the coupling parameter. When reading the literature applications of the method, it seems clear to choose this parameter to be i/k [3–29] or at least positively proportional [30, 31] or asymptotically proportional [32, 33] to this value. Some authors did not explicitely mention their choice of coupling parameter [34, 35]. Interestingly, some authors apply this value i/k as a negative value [36, 37]. Terai [38] has even clarified that the positive value is valid for a harmonic time dependence of $e^{+i\omega t}$ whereas in case of using $e^{-i\omega t}$, the coupling parameter should be negative. Especially Terai's work is essential in this context but seems to have been ignored by most authors who have used $e^{-i\omega t}$. However, Kress [14] and Amini [3] were both using this kind of harmonic time dependence and clearly found that i/k is a very good and even close to optimal choice to minimize the condition number. Apparently, it is not completely clear, what the optimal coupling parameter for the Burton and Miller formulation is. Furthermore, it will even be shown that many authors are using a coupling parameter which is not optimal.

2. BEM FORMULATION OF THE BURTON AND MILLER METHOD

Derivation of the wave equation, discussion of boundary conditions, weak formulation and discretization process are presented in a reduced way. A more detailed presentation is found in the concept chapter [39].

2.1 Helmholtz equation and Boundary Conditions

We consider linear acoustic problems defined in the domain Ω with the complement Ω_c and Γ representing the closed boundary of Ω and Ω_c. The outward normal is pointing into the complementary domain Ω_c. The wave equation

$$\Delta\tilde{p}(\vec{x},t) = \frac{1}{c^2}\,\frac{\partial^2\tilde{p}(\vec{x},t)}{\partial t^2} \qquad \vec{x}\in\Omega\subset\mathbb{R}^3 \tag{1}$$

is valid for the sound pressure $\tilde{p}$. Alternatively, a velocity potential may be used. The space dimension d is three in real applications, but can be two or one in certain cases. To complete a solution, the differential equation requires boundary conditions and initial conditions, which will be specified when used. For time–harmonic

[1] steffen.marburg-at-unibw.de

problems, a time dependence is introduced. Herein, we use the time–dependence

$$\tilde{p}(\vec{x},t) = p(\vec{x})\, \mathrm{e}^{\alpha \mathrm{i}\omega t} \tag{2}$$

with $\alpha = \pm 1$. Applying the time–harmonic dependence of p to Equation (1) leads to the Helmholtz–equation for the sound pressure.

$$\text{Helmholtz–equation:} \qquad \Delta p(\vec{x}) + k^2 p(\vec{x}) = 0 \qquad \vec{x} \in \Omega. \tag{3}$$

This result is independent of α. We assume Neumann boundary conditions for which the normal particle velocities of the fluid v_f equals the (prescribed) particle velocity of the underlying radiator v_s as

$$\text{Boundary condition:} \qquad \frac{\partial p(\vec{x})}{\partial n(\vec{x})} = skv_f(\vec{x}) = skv_s(\vec{x}) \qquad \vec{x} \in \Gamma. \tag{4}$$

The normal fluid particle velocity v_f is related to the normal derivative of the sound pressure p by means of the Euler equation in frequency domain. The wave–number $k = \omega/c$ is the quotient of the circular frequency $\omega = 2\pi f$ (f denoting frequency) and the speed of sound c; s is a constant given by $s = -\mathrm{i}\alpha\rho_0 c$. In Equation (4), i is the imaginary unit ($\mathrm{i}^2 = -1$) and ρ_0 the average density of the fluid. The vector $\vec{n}(\vec{x})$ represents the outward normal at the surface point $\vec{x}$ and $\partial/\partial n(\vec{x})$ is the normal derivative.

2.2 Weak Formulation

A weak formulation is based on introducing the weight function $\chi(\vec{x})$ and "testing" it with the Helmholtz operator such that

$$\int_\Omega \chi(\vec{x}) \left[\Delta p(\vec{x}) + k^2 p(\vec{x})\right] d\Omega(\vec{x}) \;=\; 0\,. \tag{5}$$

Integrating by parts twice gives

$$\begin{aligned} \int_\Omega \chi(\vec{x}) \left[\Delta p(\vec{x}) + k^2 p(\vec{x})\right] d\Omega(\vec{x}) \;&=\; sk \int_\Gamma \chi(\vec{x}) v_f(\vec{x}) d\Gamma(\vec{x}) + \\ &- \int_\Gamma \frac{\partial \chi(\vec{x})}{\partial n(\vec{x})} p(\vec{x}) d\Gamma(\vec{x}) + \int_\Omega p(\vec{x}) \left[\Delta\chi(\vec{x}) + k^2\chi(\vec{x})\right] d\Omega(\vec{x}) \;=\; 0\,. \end{aligned} \tag{6}$$

The second part of Equation (6) consists of one domain integral and two boundary integrals. This domain integral can be transformed into an integral–free term by using fundamental solutions $G(\vec{x},\vec{y})$ in the sense of distributions. Function G represents the solution of the equation

$$\Delta G(\vec{x},\vec{y}) + k^2 G(\vec{x},\vec{y}) \;=\; \beta\,\delta(\vec{x},\vec{y})\;, \tag{7}$$

where, similar to α in Equation (2), $\beta = \pm 1$. G is known as free–space Green's function as well, whereas $\delta(\vec{x},\vec{y})$ is the Dirac or delta function at the origin $\vec{y}$. In terms of physics, $G(\vec{x},\vec{y})$ can be understood as the sound pressure distribution according to a point source (monopole) in $\vec{y}$. Together with the harmonic time–dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$, it represents an outgoing wave. We can write G as

$$G(\vec{x},\vec{y}) \;=\; \frac{1}{4\pi} \frac{\mathrm{e}^{-\mathrm{i}\alpha k r(\vec{x},\vec{y})}}{r(\vec{x},\vec{y})} \qquad \vec{x},\vec{y} \in \mathbb{R}^3. \tag{8}$$

with r as the Euclidean distance between field point $\vec{x}$ and source point $\vec{y}$ as $r(\vec{x},\vec{y}) = |\vec{x} - \vec{y}|$. Note that the fundamental solution depends on the choice of the harmonic time dependence.

Applying the property of the fundamental solution and the delta function, we find

$$\int_\Omega p(\vec{x}) \left[\Delta G(\vec{x},\vec{y}) + k^2 G(\vec{x},\vec{y})\right] d\Omega(\vec{x}) = \int_\Omega p(\vec{x}) \beta\,\delta(\vec{x},\vec{y}) d\Omega(\vec{x}) = \beta c(\vec{y}) p(\vec{y}). \tag{9}$$

With this result and application of the boundary condition (4), Equation (6) is rewritten as

$$\beta c(\vec{y}) p(\vec{y}) + \int_\Gamma \frac{\partial G(\vec{x},\vec{y})}{\partial n(\vec{x})} p(\vec{x}) d\Gamma(\vec{x}) \;=\; sk \int_\Gamma G(\vec{x},\vec{y})\, v_s(\vec{x}) d\Gamma(\vec{x})\,. \tag{10}$$

Equation (10) is known as representation formula. For $\vec{y} \in \Gamma$, it is known as the Kirchhoff–Helmholtz (boundary) integral equation. Note that plus and minus signs of either the first term or the second and the third term may be different if the direction of the normal vector is chosen in opposite direction.

2.3 Approximation and Discretization by Collocation

Independent of the discretization method, we formulate approximations of our physical quantities. First of all, we approximate the sound pressure $p(\vec{x})$ as

$$p(\vec{x}) = \sum_{l=1}^{N} \phi_l(\vec{x})\, p_l = \phi^T(\vec{x}) p\,, \tag{11}$$

where p_l represents the discrete sound pressure at point $\vec{x}_l$ and ϕ_l is the $l-$th basis function for our approximation. Further, we assume that similar approximations are formulated for the particle velocity of the structure v_s

$$v_s(\vec{x}) = \sum_{j=1}^{\bar{N}} \bar{\phi}_j(\vec{x})\, v_{s\,j} = \bar{\phi}^T(\vec{x}) v_s \tag{12}$$

If v_s is explicitly known, these approximations are not necessary for evaluation of the boundary integrals in Equation (10). However, there are many practical cases where the structural particle velocity is the result of a finite element simulation and available only as piecewise defined function.

The number of basis functions ϕ_l and $\bar{\phi}_j$ is given by N and $\bar{N}$, respectively. If the particle velocity of the structure is a known function, N accounts for the degree of freedom. Herein, this coincides with the number of nodes of the boundary element mesh. For BEM with discontinuous boundary elements, it is common that $\bar{N} = N$.

The collocation method requires testing Equation (10) with the Dirac function $\delta(\vec{y},\vec{z})$. This integration is known analytically, cf. Equation (9). It yields

$$\beta c(\vec{z}) p(\vec{z}) + \int_\Gamma \frac{\partial G(\vec{x},\vec{z})}{\partial n(\vec{x})} p(\vec{x}) d\Gamma(\vec{x}) = sk \int_\Gamma G(\vec{x},\vec{z})\, v_s(\vec{x}) d\Gamma(\vec{x})\,, \tag{13}$$

which is basically the same expression as shown in Equation (10). The major difference between equations (10) and (13) is that the former is actually a continuous integral equation whereas the latter is valid just for the discrete point $\vec{z}$. This means that the integral equation is fulfilled at a number of discrete points, i.e. collocation points $\vec{z}_l$. It is common practice that the collocation points coincide with the nodes of the piecewise formulated approximation of the sound pressure as shown in Equation (11). For further considerations we assume that $\phi_l(\vec{z}_k) = \delta_{lk}$ where δ_{lk} is the Kronecker symbol with $\delta_{lk} = 0$ for $l \neq k$ and $\delta_{lk} = 1$ for $l = k$. Then, applying the approximation of equations (11) and (12) yields the matrix equation as

$$H p = G v_s \tag{14}$$

Matrix G is the system matrix of the single layer potential as

$$g_{l\,j} = sk \int_\Gamma G(\vec{x},\vec{z}_l)\, \bar{\phi}_j(\vec{x})\, d\Gamma(\vec{x}) \tag{15}$$

and matrix H contains the integral–free term and the contribution of the double layer potential as

$$h_{l\,j} = \beta c(\vec{z}_l)\, \delta_{l\,j} + \int_\Gamma \frac{\partial G(\vec{x},\vec{z}_l)}{\partial n(\vec{x})}\, \phi_j(\vec{x})\, d\Gamma(\vec{x})\,. \tag{16}$$

2.4 Normal Derivative Integral Equation

For further analysis, the normal derivative of equation (10) is required. This is given as

$$\beta c(\vec{y}) \frac{dp(\vec{y})}{\partial n(\vec{y})} + \int_\Gamma \frac{\partial^2 G(\vec{x},\vec{y})}{\partial n(\vec{x}) \partial n(\vec{y})} p(\vec{x}) d\Gamma(\vec{x}) = sk \int_\Gamma \frac{\partial G(\vec{x},\vec{y})}{\partial n(\vec{y})} v_s(\vec{x}) d\Gamma(\vec{x})\,. \tag{17}$$

A similar discretization process as in the previous subsection yields the matrix equation as

$$E p = F v_s \tag{18}$$

with Matrix F as the system matrix of the integral free term and the adjoint double layer potential as

$$f_{l\,j} = sk \left[-\beta c(\vec{z}_l) + \int_\Gamma \frac{\partial G(\vec{x},\vec{z}_l)}{\partial n(\vec{z}_l)}\, \bar{\phi}_j(\vec{x})\, d\Gamma(\vec{x}) \right] \tag{19}$$

and matrix E with the contribution of the hypersingular operator as

$$e_{l\,j} = \int_\Gamma \frac{\partial^2 G(\vec{x},\vec{z}_l)}{\partial n(\vec{x}) \partial n(\vec{z}_l)}\, \phi_j(\vec{x})\, d\Gamma(\vec{x})\,. \tag{20}$$

2.5 The Burton and Miller Method

The Burton and Miller method is based on a linear combination of equations (10) and (17) for which the coupling parameter η is introduced. The coupled equation reads as follows

$$\begin{aligned}\beta c(\vec{y})p(\vec{y}) \quad &+ \quad \int_\Gamma \frac{\partial G(\vec{x},\vec{y})}{\partial n(\vec{x})}\, p(\vec{x})d\Gamma(\vec{x}) + \eta \int_\Gamma \frac{\partial^2 G(\vec{x},\vec{y})}{\partial n(\vec{x})\partial n(\vec{y})}\, p(\vec{x})d\Gamma(\vec{x}) \;= \\ &= \quad sk\int_\Gamma G(\vec{x},\vec{y})\, v_s(\vec{x})d\Gamma(\vec{x}) + \eta \left[-\beta skc(\vec{y})v_s(\vec{y}) + sk\int_\Gamma \frac{\partial G(\vec{x},\vec{y})}{\partial n(\vec{y})}\, v_s(\vec{x})d\Gamma(\vec{x})\right].\end{aligned} \tag{21}$$

The matrix form is easily derived from this equation and equations (14) and (18)

$$[H+\eta E]\, p \;=\; [G+\eta F]\, v_s \tag{22}$$

which is - for the Neumann problem - solved for p.

3. TEST PROBLEMS

The formulation has been tested for different values of η in particular with respect to α and β. It turned out that identical results independent of η are evaluated if α and β are either both $+1$ or -1, hence, if the product of both is $\alpha\beta = 1$. Similarly, the results are identical for both combinations of $\alpha\beta = -1$, again independent of the choice of η. Furthermore, it turned out that the results are identical for $\eta = i/k$ in combination with $\alpha\beta = 1$ and for $\eta = -i/k$ in combination with $\alpha\beta = -1$. They are also identical for $\eta = i/k$ in combination with $\alpha\beta = -1$ and for $\eta = -i/k$ in combination with $\alpha\beta = 1$.

The different choices for α, β and η are compared for the newly generated benchmark problem of the European Acoustics Association, the Radiatterer [40]. This model of $2.5 \times 2.0 \times 1.7 m^3$ (considered in air) contains a number of resonators including a Helmholtz resonator. Linear discontinuous boundary elements are used for this evaluation [41]. When applying a unit (purely real) particle velocity to the entire Radiatterer's surface, numerous resonance peaks can be observed. The first one, a Helmholtz resonator resonance is found at approximately 20 Hz. Although very similar results are yielded for the cases of $\eta = i/k$ in combination with $\alpha\beta = 1$ and in combination with $\alpha\beta = -1$, the radiated sound power is completely different. Actually, the radiated sound power becomes negative for $\alpha\beta = -1$ whereas it is positive for $\alpha\beta = 1$. This behavior is not only observed for that particular resonance but also for other resonance peaks. It could be ruled out that this effect is neither due to integration accuracy when setting up the BE matrices nor to the accuracy of the solution of the linear system of equations. The reason for the negative sound power consists in the complex sound pressure value at these resonances. The real part of the sound pressure inside the resonator is much smaller than the imaginary part but only the real part is considered for the sound power. At the same time, the resonator region dominates the radiation. Positive and negative coupling parameters result in different signs of the (small) real part of the sound pressure inside the radiator.

The same effect is found when solving the system of equations iteratively by using a GMRes algorithm, see [42] and references therein. An $\eta = i/k$ in combination with $\alpha\beta = 1$ is almost optimal. Small adjustment to improve convergence remain. The same holds for an $\eta = -i/k$ in combination with $\alpha\beta = -1$.

Although not wrong but not optimal is it to choose $\eta = i/k$ in combination with $\alpha\beta = -1$.

Scanning the literature showed that a better coupling parameter was chosen in [3–7, 9, 12–14, 17, 22, 30–33, 36, 38] while a coupling parameter which is far from optimal was chosen in [5, 10, 11, 15, 16, 18–21, 23–29]. For some papers, it was impossible to decide whether the authors chose the better solution or not [8, 34, 35, 37]. In these cases, some information for the decision has been missing in the paper.

4. CONCLUSION

It has been interesting to see that the widely used coupling parameter of the Burton and Miller formulation is often applied in a way which is not optimal and leads to unphysical results although the numerical accuracy is still acceptable from a mathematical point of view. Interestingly, some authors (including the author of this paper) appear with both solutions in literature.

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