DESIGN OF CHIMES TO PRODUCE CONSONANT, NON-HARMONIC SCALES

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ABSTRACT : Thin, cylindrical metal chimes can be used to play in a variety of scales requiring non-harmonic spectra. The spectrum required for the chimes to have consonant intervals in a given non-harmonic scale can be determined according to a method developed by Sethares, based on the theory of dissonance proposed by Plomp and Levelt. First order perturbation theory was applied to control the frequencies of the first six modes of a thin, cylindrical metal chime by perturbations to the chime radius. The resultant chime profile was then refined using finite element methods. Two aluminium chimes manufactured with this optimised profile demonstrated superior musical performance compared with unperturbed thin cylinders.

1. INTRODUCTION

The majority of non-percussive musical instruments currently used in the West produce spectra consisting of harmonically related frequencies. That is to say, notes played using these instruments have partials that are nearly integer multiples of some fundamental frequency f_0. Such harmonic spectra are caused by a control oscillator (such as a reed, lip reed or air jet in wind instruments, or a bow in string instruments), or by the intrinsic properties of the vibrating elements used in these musical instruments, whose natural frequencies are almost exactly harmonic. Examples of the latter are long, thin, uniform, flexible strings.

When two notes produced by a harmonic musical instrument are played together, the sound produced can be either consonant, meaning pleasant and relaxed, or dissonant, meaning discordant and tense. Although there is still speculation regarding the quantitative measurement of dissonance, Plomp and Levelt [1] have proposed that the perceived dissonance between two notes is exclusively a function of the location and amplitude of the partials making up these notes. This theory of dissonance is widely accepted.

Various scales have been developed that make use of the dissonance properties of the harmonic spectrum to ensure that a fairly large range of intervals sound reasonably consonant. The 12 tone equal temperament (12-tet) scale has been widely adopted so that free modulation from one key to another is possible.

Many objects that are used, or could potentially be used to produce music, however, do not have harmonic spectra. Thus they sound discordant when played in chords using notes in a scale designed for harmonic instruments, such as the 12-tet scale. In addition, many modern composers are interested in writing music for scales other than 12-tet, which require spectra that no readily available, non-electronic musical instrument is able to produce. An approach to designing spectra for non-harmonic scales was formulated by Sethares in [2,3].

One class of non-harmonic objects with potential for wider musical applications are thin cylindrical metal chimes. In this project, a design process was developed that can be used to create a set of such chimes that have a range of consonant intervals for a given non-harmonic scale. Sethares' approach was used to determine the chime spectrum for the non-harmonic scale under consideration.

Previous work in the design of musical instruments with non-harmonic spectra includes the Pentangle by Fletcher [4]. The Pentangle consists of a pentagonal gong tuned to produce a bell-like spectrum, using a combination of analytic and finite element methods. A similar approach was followed in this study, in that analytic methods were used for basic chime design, and the chime profile was then refined using the finite element analysis package STRAND 6. Such an approach differs from the method used by Petrolito and Legge [5], who applied finite element methods alone to tune the first three modes of xylophone bars with rectangular cross sections. Thus while the approach adopted by Petrolito and Legge was to constrain the finite element optimisation process using criteria such as the minimisation of material to be removed, the approach of this work and of Fletcher was to limit the finite element based optimisation to solutions close to results obtained using simplified theoretical models of the system. As far as we are aware, however, this study is the first attempt to produce a non-electronic musical instrument that implements Sethares' method of selecting a spectrum for a scale.

2. CHIME FREQUENCY SHIFT CALCULATION USING PERTURBATION THEORY

The theoretical model that was used to obtain a chime profile that produces a close approximation to the desired spectrum is described. Equations are given for the bending modes of a thin, cylindrical rod, and then a general theory is introduced to describe the frequency shifts caused by perturbations to a vibrating object. Next, this theory is used to predict the shifts to rod frequencies when small changes are made to the radius of a rod.

Finally, this theory is applied to the specific problem of controlling the frequencies of a cylindrical metal chime. The problem of achieving the chime goal frequencies while minimising the changes to chime radius is also addressed, since the perturbation theory predictions will only be accurate for small changes to the rod profile.

2.1 Bending wave equation for a rod

The general wave equation for bending waves in a uniform rod is [6]

$$\frac{\partial^2}{\partial x^2}\left(-ES\kappa^2\frac{\partial^2\Psi}{\partial x^2}\right)=\rho S\frac{\partial^2\Psi}{\partial t^2}\,, \qquad (1)$$

where E is the Young's modulus, S is the cross sectional area ($S=\pi r^2$ for a cylindrical rod), κ is the radius of gyration about the neutral axis ($\kappa = r/2$ for a cylindrical rod), ρ is the density, x is the position along the rod as measured from the middle, t is time, and $\Psi(x,t)$ is the displacement of the rod from equilibrium.

Equation (1) is a "thin beam" approximation that neglects such effects as shear and rotary inertia, and only models bending modes. For this approximation to be true, the wavelength of the modes must be large compared with the rod diameter, since otherwise shear distortion is non-negligible in comparison with bending. Since we only consider the first six modes of the rod (the most audible modes), this translates as a requirement that the length to dimeter ratio for the rod should be around 60:1 or greater. In practice, this requirement may be relaxed a little.

The solutions to (1) take the form of sums of members of the series

$$\Psi_n=\psi_n(x)\cos(\omega_n t), \qquad (2)$$

where

$$\psi_n = A_n\cosh\frac{\omega_n x}{v_n}+C_n\cos\frac{\omega_n x}{v_n}+B_n\sinh\frac{\omega_n x}{v_n}+D_n\sin\frac{\omega_n x}{v_n} \qquad (3)$$

Here $v_n=\left(\omega_n^2\kappa^2 e/\rho\right)^{1/4}$ is the wave velocity of mode n with angular frequency ω_n, and A_n, B_n, C_n and D_n are constants.

For a rod with both ends free, the solutions to equation (1) have $A_n=C_n=0$ for even n and $B_n=D_n=0$ for odd n, since ψ_n is even for odd n and odd for even n. Bending oscillations in a rod of constant radius with both ends free occur at frequencies f_n, given by

$$f_n\approx\frac{\pi\kappa}{8L^2}\sqrt{\frac{E}{\rho}}(2n+1)^2 \qquad (4)$$

2.2 Frequency shifts to a cylindrical rod due to changes in rod radius

A general perturbation theory that can be used to determine the effect on rod frequencies of perturbations to the rod radius is now described. This theory takes a form similar to that proposed by Tse, Morse and Hinkle [7]. Only first order perturbations will be taken into account, which means that the frequency shifts predicted are approximate. As a starting point, it is possible to write (1) in the form

$$-\omega_n^2\psi_n=\Im\psi_n\,, \qquad (5)$$

where $\Im$ is a linear differential operator, given by

$$\Im=-\frac{E}{\rho}\cdot\frac{1}{S}\cdot\frac{\partial^2}{\partial x^2}\left(S\kappa^2\frac{\partial^2}{\partial x^2}\right) \qquad (6)$$

for the rod, and we take the functions ψ_n to be normalised so that $\int_{-L/2}^{L/2}\psi_n^2 dx=1$. We now consider the effect of making the perturbations

$$\omega_n^2\to\omega_n^2+\delta\left(\omega_n^2\right),\quad \Im\to\Im+\delta\Im,\quad \psi_n\to\psi_n+\delta\psi_n \qquad (7)$$

in (5). Expanding (5), and retaining only first order or lower terms, gives

$$\frac{\delta\omega_n}{\omega_n}=-1+\sqrt{1-\omega_n^{-2}\int_{-L/2}^{L/2}\psi_n\delta\Im\psi_n\,dx}=-1+\sqrt{1-\zeta_n} \qquad (8)$$

where we have written $\zeta_n=\omega_n^{-2}\int_{-L/2}^{L/2}\psi_n\delta\Im\psi_n dx$ for ease of reference.

We now derive a formula for the ζ_n due to perturbations in the rod radius. The expression for $\Im$ in the case of the rod is given by (6). Making the changes $r\to r+\delta r(x)$ and $\Im\to\Im+\delta\Im$ in (6), retaining only first order and lower terms in the resulting expansion, and then substituting this expression for $\delta\Im$ into (8), gives

$$\zeta_n=\int_{-L/2}^{L/2}\left(\frac{\delta r(x)}{r}\psi_n^2\right)dx-\frac{32L^4}{\pi^4 Q_n}\int_{-L/2}^{L/2}\left[\frac{\partial^2}{\partial x^2}\left(\frac{\delta r(x)}{r}\frac{\partial^2(\psi_n)}{\partial x^2}\right)\right]\psi_n dx, \qquad (9)$$

where

$$Q_n\approx(2n+1)^4 \qquad (10)$$

2.3 Design of a chime with a specified set of frequencies

Equations (9) and (8) may now be used to obtain a cylindrical metal chime whose frequencies of bending oscillation can be assigned specified values. Given that N mode frequencies are to be controlled, the perturbation to the chime radius will take the form of a sum of N terms,

$$\delta r(x)=\sum_{m=1}^{N}b_m\delta r_m(x) \qquad (11)$$

where the $\delta r_m(x)$ represent perturbations to the chime radius as functions of longitudinal coordinate x, and each constant b_m scales the perturbation m. A set of functions for the $\delta r_m(x)$ is now chosen, with the intention that each perturbing function should have as large an effect as possible on only one modal frequency, and a minimum effect on all other frequencies.

For a given increase in chime radius δr, the stiffness will increase by a factor $1+4\delta r$ while the mass will increase by $1+2\delta r$. In addition, for all modes, regions of high lateral displacement from equilibrium correspond to regions of large curvature over most of the chime (the exception being near the ends, where the displacement is significant but the curvature approaches zero). Therefore if the chime radius is increased in regions where a given mode has a large curvature, then the frequency increase for this mode due to the higher stiffness will be greater than the drop in frequency due to the increased mass, and the net effect will be an increase in mode frequency.

From this property, it seems likely that the perturbing functions $\delta r_m(x)$ will have larger effects on unique mode frequencies if each perturbing function only has a large amplitude where the mode that it targets has large curvature. Thus the set of perturbing functions adopted is

$$\delta r_m(x) = \left(\frac{\partial^2 \psi_m}{\partial x^2}\right)^2 - \frac{1}{L}\int_{-L/2}^{L/2} \left(\frac{\partial^2 \psi_m}{\partial x^2}\right)^2 dx \qquad (12)$$

The purpose of the integral is to avoid applying an upwards shift on all modes when the radius is perturbed. Substituting (11) into (9) allows (9) to be rewritten as

$$\zeta_n = \sum_{m=1}^{N} b_m U_{n,m}, \qquad (13)$$

where

$$U_{n,m} = \int_{-L/2}^{L/2} \left(\frac{\delta r_m(x)}{r}\psi_n^2\right) dx - \frac{32L^4}{\pi^4 Q_n}\int_{-L/2}^{L/2} \left[\frac{\partial^2}{\partial x^2}\left(\frac{\delta r_m(x)}{r}\frac{\partial^2 \psi_n}{\partial x^2}\right)\right]\psi_n dx \qquad (14)$$

Once the N desired frequency shifts $\delta\omega_n$ have been specified, (8) may be solved to obtain the N quantities ζ_n. The set of linear equations (13) can then be solved to give the amplitudes of the perturbing functions in (11).

3. CHOOSING A TARGET SPECTRUM

Now that an approximate process for controlling the frequencies of a cylindrical chime has been illustrated, the chime goal spectrum that was used to test this procedure is described. Given that the object of this project was to find a design process that can be used to create chimes for a given scale, we decided to test this process by selecting a genuinely arbitrary scale and working towards a set of chimes with appropriate spectra.

The scale that was chosen for the chimes is an unequally stretched, Pythagorean major scale. The unstretched Pythagorean major scale sounds quite similar to the 12-tet major scale, but with slightly sharper major thirds and slightly flatter minor thirds [3]. The steps of the Pythagorean scale, whether stretched or unstretched, can be written

$$\left(1,\ a,\ a^2,\ a^2b,\ a^3b,\ a^4b,\ a^5b,\ a^5b^2\right)\times f_0\ , \qquad (15)$$

where f_0 represents the frequency of the first note in the scale and a and b are the two basic frequency step ratios. The notation in (15) means that the first note in the scale has frequency $1\times f_0$, the second note has frequency $a\times f_0$, and so on. The unstretched Pythagorean scale has a=9/8 and b=256/243. Our stretched scale had a=6/5 and b=16/15. This means that the "octave" (literally the eighth note) occurs at 1,990,656 / 703,125 ≈ 2.83 times the frequency of the lowest note in this scale, rather than the usual factor of 2 for unstretched scales. For this reason, the term *pseudo-octave* will be used to designate the stretched "octave" interval.

One possible chime spectrum for the scale described in (15), as determined using the procedure outlined by Sethares [2,3], is

$$T = \left(1,\ a^5b^2,\ a^8b^3,\ a^{11}b^4,\ a^{13}b^5,\ a^{15}b^5\right)\times f_0\ , \qquad (16)$$

where the notation of equation (15) has been used, and the number of partials is $N = 6$. The required frequency shifts to the modes of the unperturbed chime, as fractions of the unperturbed mode frequencies, vary from -8% to 9%.

To see why the target spectrum in equation (16) represents a good spectrum for the scale in (15), it is necessary to introduce the concept of the *dissonance curve* for a spectrum. This is a plot of the dissonance (as defined by Plomp and Levelt [1]) perceived when two notes are played simultaneously, where the frequencies of the partials of the higher note are increased by a factor R with respect to the partials of the lower note. Plomp and Levelt found that when these two notes are pure sinusoids, the dissonance $d(a_1, a_2, f_1, f_2)$ between them is a function of the amplitudes a_1 and a_2 of the sinusoids, and their frequencies f_1 and f_2. The dissonance between the sinusoids is zero when their frequencies are equal, rises rapidly as the frequency difference increases, and then drops slowly upon further increase in the frequency difference. A plot of the dissonance $d(a_1, a_2, f_1, f_2)$ between two sinusoids is given in Figure 1, where $a_1 = a_2 = 1$ and $f_2 = R \times f_1$. Plomp and Levelt found that this function $d(a_1, a_2, f_1, f_2)$ closely approximated the average dissonance measured in a series of tests involving 90 musically untrained volunteers.

The dissonance $g(R)$ of a spectrum containing partials at several frequencies is obtained by adding the dissonance of every pair of partials, so that

$$g(R) = \sum_{i=1}^{N}\sum_{j=1}^{N} d(f_i, Rf_j, a_i, a_j)\ , \qquad (17)$$

where a_i and a_j are the amplitudes of the partials with frequencies f_i and f_j respectively. The dissonance curve for a given spectrum is a plot of $g(R)$ for that spectrum.

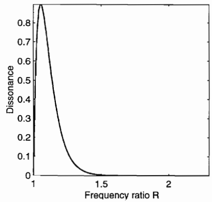

Figure 1 : dissonance between two sinusoids with frequencies f_1 and f_2, where $f_2 = R \times f_1$. The sinusoids have equal amplitude.

The dissonance curve for the spectrum in (16) is illustrated in Figure 2, where stems are used to show the location of the steps of the scale described in (15). We see from this plot that dissonance is low at most of the scale steps, indicating that chords played using these intervals will sound pleasant.

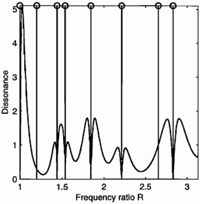

Figure 2 : Dissonance curve of a the stretched Pythagorean scale, a=6/5, b=16/15, for the spectrum defined in (16). The stems represent the intervals defined in (15). Amplitudes of all partials are assumed equal to one.

4. DESIGN OF THE CHIME

The approximate chime design procedure described in section 2 was applied to design a chime to play in the goal spectrum. A finite element model of the chime was used as a fast and inexpensive means of testing and refining the results obtained using perturbation theory. The finite element model was formulated using the STRAND 6 finite element analysis package.

4.1 Chime profile obtained using perturbation theory

The profile for a chime that produces the spectrum given by (16) was generated in accordance with section 2.3. The unperturbed cylinder on which the chime was based has length L=0.5215m and diameter 6.02mm. These dimensions satisfy the thin rod approximation for the first six modes as described in section 2.1. The six perturbing functions for the chime, determined in accordance with equation (12) of section 2.3, are illustrated in Figure 3.

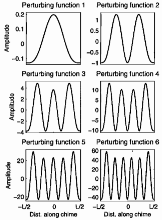

Figure 3 : The six perturbing functions for the chime.

The solid profile in Figure 4 illustrates the chime obtained by summing the scaled perturbing functions and adding the result to the unperturbed cylindrical chime. It was not possible to manufacture the smooth profile illustrated in Figure 4, since the chimes were too long and thin to be machinable using a computer controlled lathe, and the sinusoidal shape was too difficult to reproduce by hand. For this reason, a step approximation to the profile, as illustrated by the dashed curve in Figure 4, was designed. Although this step approximation caused slight shifts to the goal frequencies, these remained small (0.7% of the original goal frequencies at most, when calculated using perturbation theory), and were in any case reduced by the refinement process to be described in the following section.

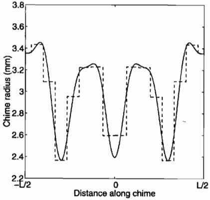

Figure 4 : Profile obtained using first order perturbation theory, and manufacturable step approximation to this profile. ——— : ideal profile. .... : step profile.

4.2 Refinement of chime profile using finite element methods

The finite element analysis package STRAND 6 was used to model the chime shown in the step approximation from Figure 4. The natural frequencies calculated by the finite element model for this chime profile differed from the goal frequencies by amounts ranging from 2.0% to -2.4%. of the goal frequencies. An improvement in the musical performance of the chimes, however, is deemed to occur when the differences between the goal frequencies and the STRAND 6 simulation frequencies drop below 1%, since our experience (based on a computer simulation of "chime-like" chords) was that perceptible improvements in consonance occur once the error in the partial frequencies falls below 1%.

For this reason, an optimisation process was undertaken using STRAND 6. Small changes to the profile of the finite element model were made, and the changes to the natural frequencies calculated by the model were observed. This data was used to make small changes to the chime profile that improved the agreement of the calculated chime frequencies with the goal frequencies. Simulation of the optimised profile in STRAND 6 resulted in frequency differences between the simulation and goal frequencies of 0.89% of the goal frequencies or less, which is below the level of 1% mentioned above. The optimised profile is illustrated in Figure 5.

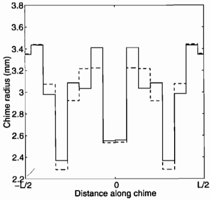

Figure 5 : Chime profile obtained using STRAND 6 based optimisation vs profile obtained using perturbation theory. ——: STRAND 6 optimised. ......: perturbation theory only.

5. MANUFACTURING AND TESTING THE CHIMES

The chime design arrived at using STRAND 6 optimisation was tested experimentally by manufacturing two chimes, and measuring their frequencies of oscillation. The results of these measurements are summarised, and explanations are proposed for the differences between the measured results and the STRAND 6 frequencies.

5.1 Chime dimension selection

Since funds were only available to make two chimes, these chimes were designed so that one plays a "pseudo-octave" above the other. The reason for this choice is that this interval represents the limits of the scale in which the chimes are to be played. Thus the interval gives a good indication of whether these limits are attainable using the chimes. Another advantage of using this interval is that unperturbed chimes sound dissonant when played in the same interval. Finally, the octave is usually judged the most consonant interval in the commonly used temperaments, and so it is especially interesting to improve its consonance in new timbre/tuning combinations.

A fundamental of around 170Hz was chosen for the chime that plays the tonic in the interval. This ensures that the first six partials of both chimes are audible. When scaling the chimes to produce these two notes, a constant average radius was maintained and only the chime lengths were varied. This causes the radiated sound volume from both chimes to be fairly similar.

A disadvantage of this scaling method is that the length to diameter ratio for the octave chime is at most 35:1, which is below the ratio required for the thin rod approximation to hold for the fourth to sixth modes. The frequencies predicted by STRAND 6 for these modes, especially for the fifth and sixth modes, may therefore be inaccurate. This effect could be countered using thinner chimes, however such chimes would become too difficult to machine. The dimensions of the two chimes manufactured are given in Table 1. Both chimes were turned from an aluminium rod using a manually controlled lathe.

Table 1 : Dimensions of the tonic and pseudo-octave chimes

Dimension	Tonic chime	Octave chime
Length L (mm)	372.8	221.7
Mean diameter (mm)	5.83	5.83

5.2 Measurement of manufactured chime frequencies, comparison with STRAND 6 model.

The chime frequencies were measured using a microphone attached to a computer. Sound from the microphone output was sampled for 2 seconds at 44.1kHz, and the frequency spectrum was obtained by taking the FFT of this signal. Chimes were suspended using cotton to simulate the boundary conditions (the chime must be free at both ends).

Table 2 contains a comparison between the spectra of the manufactured chimes and the goal spectrum in equation (16). The partials of the goal spectrum are expressed as ratios with respect to the goal fundamental. The measured partial frequencies are expressed as ratios of the scaled fundamental f_s, which is equal to

$$f_s = Nf\left(\sum_{n=1}^{N} r_{G,n}/r_{M,n}\right)^{-1}, \tag{18}$$

where f is the measured fundamental frequency, $r_{G,n}$ is the ratio of the nth goal frequency to the goal fundamental, and $r_{M,n}$ is the ratio of the nth measured frequency to the measured fundamental. This scaled fundamental is used so that the measured frequencies are compared with the pitch centroid of the goal frequencies.

Table 2 : Results of the measurement of tonic and pseudo-octave chime frequencies

Mode	Goalfrequency ratio with respect to fundamental frequency	Measured frequency ratio with respect to scaled fundamental frequency	
		Tonic	Pseudo-octave
1	1.00	1.01	1.02
2	2.83	2.82	2.85
3	5.22	5.26	5.29
4	9.62	9.58	9.58
5	14.77	14.84	14.73
6	21.27	20.91	20.63

As was noted previously, a significant reduction in dissonance occurs once the errors in the partial frequencies drop below 1% (approximately 1/6th of a semitone in the 12-tet scale). In the right hand column of Table 2, we note that the errors in the ratios of four of the partials for the chime at the tonic, and three of the partials for the chime at the pseudo-octave, are less than 1%. Thus consonance will be improved for at least some of the chords playable in the scale proposed in (15), when comparing chimes manufactured with the STRAND 6 optimised profile to unperturbed chimes.

There are a number of factors contributing to the errors in the STRAND 6 predictions of the perturbed chime frequencies. These include differences in segment length and chime radius when comparing the manufactured chimes to the finite element model, and the uncertainty in the location of the measured partials due to the resolution of the recording equipment. These effects, however, are less important than the consequences of inaccuracies in our finite element model. The STRAND 6 simulation used to predict the chime frequencies made use of a thin rod approximation that neglects shear effects, as well as being slightly inaccurate in its treatment of the forces between adjacent cylindrical segments. These effects cause our simulation to yield higher frequencies than expected for higher modes, hence the downward trend in the measured frequency ratios when comparing to the goal frequency ratios.

6. CONCLUSION

The results obtained in this study are encouraging, since they indicate that the musical performance of chimes designed using the method proposed is perceptibly superior to the performance of unperturbed cylindrical rods, for the stretched Pythagorean scale. Error with respect to the goal spectrum in the partials of the manufactured chimes was mainly due to shear effects not modelled by our finite element simulation, and/or the incorrect assumptions made in this simulation when joining thin beam elements end to end.

There are a number of ways to further improve the performance of these chimes. The chimes could be tuned by hand, or by changing the goal frequencies used in the finite element optimisation of the profile derived using perturbation theory. Another approach would be to use a more sophisticated finite element model of the chimes. Alternatively, the chime profile could be designed by perturbing the Timoshenko beam equations [7], which take shear effects into account and do not require a thin beam approximation to hold. This would allow the diameter of the chimes to be greatly increased, thus increasing both manufacturability and acoustic radiation efficiency. Once a single such chime has been produced with a satisfactory spectrum, a set of chimes could easily be manufactured by linearly scaling the length and diameter of the successful design.

Perhaps the most exciting outcome of this research is that these chimes represent, to our knowledge, the first attempt at applying Sethares' method [2,3] of creating a spectrum for a musical scale in a non-electronic instrument. Musical instrument design methods such as the one proposed in this study have the possibility of opening up vast new realms of musical potential to composers. The evolution of these design methods, both for cylindrical chimes and for other non-harmonic instruments, will undoubtedly be a fascinating process.

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