GRAVITATIONAL OSCILLATORS: BOUNCING BALLS, ROCKING BEAMS, AND SPINNING DISCS

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When an object is constrained to lie in a half-space bounded by a rigid horizontal wall, it will fall against this wall under the influence of gravity and then rebound in some way so as to execute oscillations which will gradually decay as energy is lost in the collisions. Common cases are bouncing balls, rocking beams, and discs allowed to spin obliquely onto the surface. In this last case, exemplified by coins and saucepan lids, the resulting radiated sound has interesting properties.

1. INTRODUCTION

Upon reflection, we are all familiar with the behaviour of a disc such as a coin or a saucepan lid dropped obliquely upon a table. After a brief unstable wobble it settles into a controlled motion in which its tilt rotates rapidly in either a clockwise or anticlockwise direction and any pattern on the surface is seen to rotate very slowly in the same direction. As the mechanical energy dies away over a few seconds the rotation rate of the slope angle speeds up but that of the surface pattern slows down. This behaviour is coupled to the sound radiated from the disc, so that we hear a rather broadband sound, the frequency of which rises at first slowly and then rapidly as the disc settles to the table surface.

The physical principles underlying the behaviour of such an object, known as “Euler’s disc”, have been treated in several classical texts [1,2] but still attract attention even in journals as prominent as Nature [4,5]. My aim in the present short paper is to explain the behaviour of such a disc, to link it to that of other simpler constrained oscillators such as bounding balls and rocking beams, and then to examine the resulting acoustic excitation and radiation.

2. BOUNCING BALLS

The simplest case to consider is that of an elastic ball dropped onto a flat surface. The ball rebounds, but its rebound velocity is less than its impact velocity by a factor $\alpha<1$ known as the coefficient of restitution. The rebound energy, and thus the rebound height, is therefore reduced by a factor α^2. If the rebound velocity is v, then cycle time until the next impact is $2v/g$, where g is the acceleration due to gravity. After n bounces, the bounce height has been reduced by a factor α^{2n} and the cycle time by a factor α^n. The impact frequency f_n after n rebounds has the value

$$f_n = \frac{g}{2v_0\alpha^n} = \left(\frac{g}{8h_n}\right)^{1/2} \tag{1}$$

where v_0 is the initial release velocity and h_n is the bounce height after the nth bounce. From this it can be deduced, after a little algebra, that the oscillation actually ceases at a time t_∞ after the first impact, where

$$t_\infty = \frac{2v_0\alpha}{g(1-\alpha)} = \left(\frac{8h_0}{g}\right)^{1/2}\left(\frac{\alpha}{1-\alpha}\right) \tag{2}$$

in which h_0 is the initial release height. More importantly from our present viewpoint, the impact frequency varies as a function of time according to the equation

$$f(t) = \frac{\alpha g}{2\alpha v_0 - gt(1-\alpha)} \ . \tag{3}$$

This result is shown in Figure 1, from which it is clear that the impact frequency rises at an increasing rate and actually becomes formally infinite just before the oscillation ceases at time t_∞. We shall see that some of these behavioural features apply at least qualitatively to all the gravitational oscillators we discuss. A simple experiment with a super-elastic ball verifies the predictions of the model.

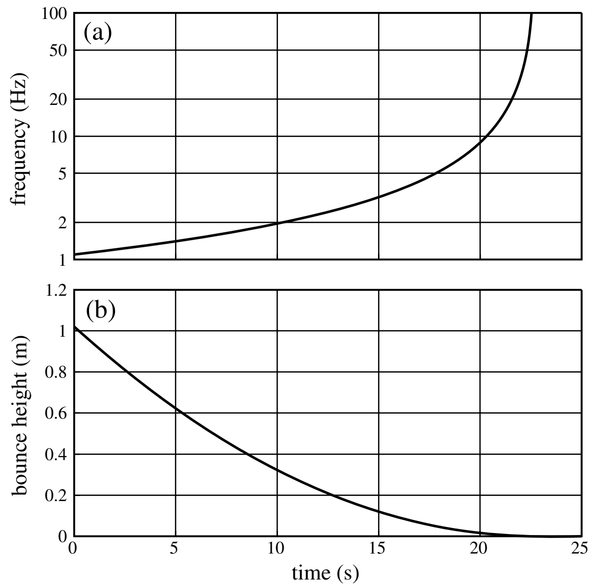

Figure 1: Calculated behaviour of (a) bounce frequency and (b) bounce height as functions of time after initial release for an elastic ball released from a height of 1m. Assumed coefficient of restitution α is 0.96 and the value of t_∞ is then 21.7s.

3. ROCKING BEAMS

We turn now to a rather more complex system consisting of a beam rocking on two symmetrically placed supporting ridges, as shown in Figure 2(c). To simplify the analysis the solid beam can be replaced by two point masses located symmetrically on a light beam as in (a) and (b), the beam also being symmetrically located relative to the two fulcra. As an initial condition we suppose that the beam is in contact with just one fulcrum A, as in Figure 2(a), and is released from a stationary state inclined at an angle. It is assumed that the beam does not slide on the fulcrum and remains in contact until the beam impacts on the other fulcrum B. There are now impulsive vertical forces acting on the beam at its two points of contact and these serve both to reverse the sign of the motion of the centre of mass and also to change the speed of rotation of the bar. Just what occurs then depends upon the ratio of the distance between the two fulcra and the distance between the two masses.

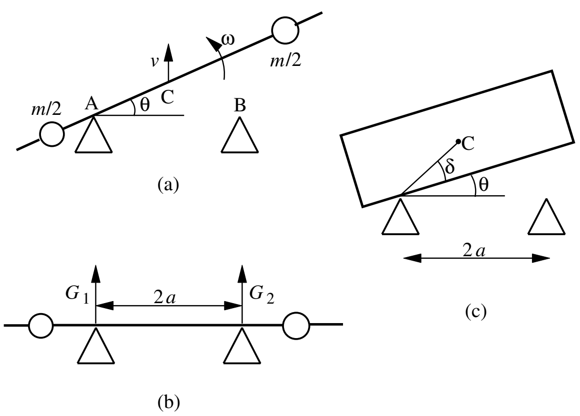

Figure 2: Simplified geometry of a beam rocking upon two symmetrically placed supports. When the beam is in contact with only one support, as in (a), there is a non-impulsive force acting upon it, but when it contacts both supports, as in (b), there are impulsive forces G_1 and G_2 acting as well. Panel (c) shows a more realistic situation for a beam of finite thickness.

Analysis of the behaviour is straightforward but the results are simple and interesting. Suppose that the mass of the beam and its attached masses is m, that its moment of inertia about the centre of mass is I, and that the distance between the two supporting fulcra is $2a$. Suppose also that the impacts are ideally elastic so that there is no loss of energy and let v_1 and ω_1 be the centre of mass velocity and the rotation velocity respectively just before impact, with v_2 and ω_2 being the same quantities just after impact. Then the impact equations can be solved to give

$$v_2 = \left(\frac{ma^2 - 3I}{I + ma^2} \right) v_1 \tag{4}$$

$$\omega_2 = \left(\frac{I - 3ma^2}{I + ma^2} \right) \omega_1 . \tag{5}$$

Several simple cases arise. The first occurs if $I = ma^2$, which is the case for two simple masses on a light rod and separated by the same distance as separates the two fulcra, or for a uniform rod of length $a\sqrt{12} \approx 3.46a$. For these parameter values the motion simply reverses with $v_2 = -v_1$ and $\omega_2 = -\omega_1$ and there is no impulse on fulcrum A. This is the equivalent of finding the "sweet spot" for hitting a ball with a cricket bat or tennis racquet. The other simple case occurs if $I = 3ma^2$, which can be achieved by a uniform beam of length $6a$, for then $v_2 = -2v_1$ and $\omega_2 = 0$ so that the beam loses contact with both fulcra and simply bounces vertically, maintaining its horizontal orientation. The beam can then continue to bounce up and down, remaining horizontal. The only other really simple behaviour is that which occurs for the limiting case in which the beam length, or separation between the two masses in the simple case, is almost infinitely long compared with the fulcrum separation $2a$. In this case support of the beam is simply transferred from one fulcrum to the other and the beam continues to rotate in the same direction at the same angular speed until gravity causes the motion to reverse. In all other cases, and particularly if $ma^2 < I < 3ma^2$, the motion involves bouncing contact on both the fulcra immediately after the initial impact. Allowing a certain amount of energy loss upon impact blurs the distinctions a little, so that there is a small region around each parameter value in which its distinctive behaviour can be expressed.

There is one other significant result that emerges from the analysis, and this relates to the frequency of the rocking oscillations. The equations are complicated but, for the long-beam case in which contact is transferred repeatedly from one fulcrum to the other, the result is that for oscillations of small amplitude θ

$$f \approx \frac{1}{2} \left[\frac{mga^2}{2h(I + ma^2)} \right]^{1/2} , \tag{6}$$

where $h = a \sin\theta$ is the maximum height reached by the centre of mass. This is qualitatively similar to the result found for a bouncing ball in equation (1), with $f \propto h^{-1/2}$, so that the evolution of the oscillation frequency will follow essentially the same path as that shown in Figure 1(a) if there are fractional energy losses at each impact.

4. SPINNING DISCS

After this preamble we come now to the main topic of this paper, the behaviour of a disc released at an angle onto a horizontal plane in such a way that it begins a spinning motion. This could also be achieved either by initially rolling the disc along on its edge or by spinning it with its plane vertical. The disc will then collapse into the behaviour to be considered here. As noted in Section 1, the angle at which the disc is inclined to the support plane, and hence the point of contact, is observed to precess rapidly, while the disc itself rotates only slowly in the same angular sense. We will now examine the processes by which this comes about and the consequences for vibration and sound radiation from the structures involved.

The physical situation and coordinates involved are both shown in Figure 3. We take the radius of the disc to be a and that of the contact circle to be r, the angular speed of rotation of the disc about its axis to be ω and the rate of precession of the contact point on the plane to be Ω. Then the condition that there be no slipping at the contact point requires that $r\Omega = -a\omega$. The apparent rate of rotation of the pattern on the upper face of the disc is $\Omega+\omega = \Omega(1-r/a)$, and this is observed to have the same direction as that of the rotation of the contact point, which implies that $r < a$.

Understandably, analysis of the motion of the disc is rather complex, though essentially straightforward, since two coupled rotating motions must be considered as well as the stability of the height of the centre of mass above the plane. There are two ways of approaching this calculation. The first, and more usually adopted, is based upon moments and rotational inertia, while the second splits the motion into two sinusoidal oscillations that are geometrically at right angles and temporally 90 degrees out of phase. The final result is the same in both cases and gives the rotational angular velocity of the point of contact of the disc on the supporting plane as

$$\Omega = \left[\frac{4g\cos\theta}{(6\delta + a\cos\theta)\sin\theta}\right]^{1/2} \qquad (7)$$

where $\delta = r - a\cos\theta$ is the radius of the circle traced out by the centre of mass of the disc. This is the result given by Ramsey [1], while Olsson [3] adds the assumption that $\delta = 0$. In reality it seems that δ may vary with the initial conditions, since for a disc spinning in a vertical plane $\delta = 0$ initially, while for a vertical rolling disc $\delta = \infty$. When the disc motion collapses to its inclined-plane state, however, it seems that δ converges towards a standard value near zero. Details have not yet been worked out.

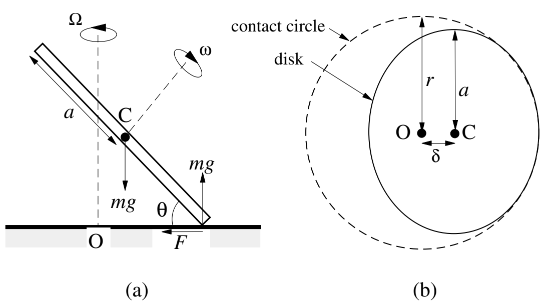

Figure 3: (a) Elevation view of the precessing disc, showing the forces acting upon it, F being a frictional force. (b) Plan view. C is the centre of the disc and O the centre of its circular contact path with the supporting plane, shown as a broken line. F is a frictional force that prevents the disc from slipping.

If δ is very much less than the disc radius a, then equation (7) shows that the angular rate of rotation Ω of the contact point between the disc and the plane becomes

$$\Omega \approx \left(\frac{4g}{h}\right)^{1/2}, \qquad (8)$$

where h is the height of the disc centre above the plane, while the visual rotation rate of the pattern on the disc becomes

$$\Omega + \omega \approx \frac{h^2}{2a^2}\left(\frac{4g}{h}\right)^{1/2}. \qquad (9)$$

If energy is lost, then from (8) the rotation rate of the disc increases as $h^{-1/2}$, which is just the same behaviour found before in the case of bouncing balls or rocking beams and illustrated in Figure 1. The visual rotation rate of the pattern, however, does not increase but rather, as shown in (9), decreases as $h^{3/2}$. Both these effects can be easily observed for a spinning coin or saucepan lid.

5. ACOUSTIC EFFECTS

We come now to examine the vibrational interaction between the disc and the supporting plane and the way in which sound is generated and radiated. These matters are simple in the case of a bouncing ball or a rocking beam, since the excitation consists of a series of impulses delivered to both the moving object and the supporting plane at a rate that increases with time in the manner illustrated in Figure 1. For a spinning disc, however, there is simply a constant force mg applied to the disc rim at a point, the position of which rotates at a speed given by equation (8).

If the supporting plane is very large so that reflections from its edges can be neglected, then radiated sound comes simply from the vibrational waves propagating away from the rotating source. This gives a wave equation on the plane of the form

$$\rho\frac{\partial^2 z}{\partial t^2} + S\nabla^4 z = mg\delta(x - r\cos\omega t)\delta(y - r\sin\omega t), \qquad (10)$$

where z is the displacement normal to the supporting plane, S is the elastic stiffness of that plane, and $\delta(x-x_0)$ is a Dirac delta function. Formal solution of this equation is not simple, but it clearly leads to elastic waves propagating outwards from the place where the rotating disc is located. These waves will not be exactly sinusoidal because of an effective Doppler shift as the contact point moves around the circle.

In the case of a coin, there is another matter to consider and this is the effect of the milled edge, which adds a high harmonic to the excitation. A coin typically has about two milled grooves per millimetre on its circumference, which typically measures between 50 and 100 mm, so that we are dealing with an excitation frequency that is 100 to 200 times the coin rotation frequency, which brings it into the low kilohertz range. However, since the duration of the rotation phase for a coin is brief, it is quite likely that it will spend much of this time in a transient state in which there may even be rebounds from the surface, which will add impulsive excitations. A much larger disc, such as a saucepan lid, can be manipulated more easily into a stable rotating state and also maintains this state for much longer, so that it is easier to observe and hear.

Excitation of the disc itself is quite different, since it is a confined structure and generally has well-defined normal modes of the form $\psi_{mn} = R_{mn}(r)\cos n\phi$, or the sine equivalent, where ψ_{mn} has m nodal circles and n nodal diameters. Since the excitation point applies a force mg and is moving round the free circumference of the disc with angular speed Ω, it effectively applies an excitation force of frequency $n\Omega$ to mode (m,n), so that the excitation is effective at quite a high frequency. While it would be possible to analyse this behaviour in detail, the problem is hardly of sufficient importance to warrant this. This excitation signal sweeps upwards with time as the disc loses rotational energy and its centre of mass moves closer to the plane, as described by equation (8), and this is clearly audible.

Sound radiation from the rotating disc is complicated for several reasons. The oscillating disc is itself a set of multipole sources, each of distinct order mn, which are all correlated in phase at the contact point. The lower side of the disc, however, is shielded by the supporting plane and its oscillations are imposed upon the wedge of air between the disc and the plane, from where they radiate preferentially in a direction opposite to the contact point. This imposes a rapid fluctuation on the sound in any given direction.

There is one other interesting resonance phenomenon that influences the sound radiation from a disc. This comes from the fact that the air volume enclosed under the disc, and vented by the opening between it and the supporting plane, acts as a resonator, which may impose a sort of "vocal formant" on the radiated sound. In the case of a domed disc, such as a saucepan lid, the resonance frequency f^* is determined by the enclosed air volume V and the area S of the vent. In such cases we can easily deduce the approximate relation

$$f^* \approx \frac{c}{2\pi}\left(\frac{S}{Vd}\right)^{1/2} \tag{11}$$

where c is the velocity of sound and d is the radiation "end correction" applicable to the opening. This Helmholtz approximation is, however, valid only if the sound wavelength at frequency f^* is large compared with the diameter of the disc. For a domed disc such as found in a saucepan lid, the enclosed volume has the form $V_0 + \alpha h$ where V_0 is the volume of the dome itself and α is a constant. If $\alpha h << V_0$, as will be the case when the disc has nearly settled, then since S is also proportional to h while d tends to a small constant value, (11) predicts that f will be about proportional to $h^{1/2}$ so that the formant frequency will decrease as the disc sinks towards the plane.

The situation for a plane disc is rather different, since the geometry of the air volume is simply reduced in the vertical direction as the disc settles, and this is rather analogous to changing the divergence angle of a conical horn, which has almost no effect on its resonance frequency. We should expect this frequency to be a little less than $c/2D$ where D is the diameter of the disc.

The behaviour of a domed lid rotating and settling over a period of about 5 seconds is illustrated in the sonogram of Figure 4. The lid itself was 13 cm in diameter and about 30 mm in height at its centre, giving an enclosed volume of about 300 cm^3. Its material was steel about 0.3 mm in thickness. The sound intensity occurs as pulses as the raised side of the disk rotates, initially at a frequency of about 7 Hz but then increasingly rapidly. The lower dark band at about 800 Hz is the formant described above, and it is clear that its frequency reduces slowly over most of the time before plunging fairly abruptly to zero in the final few tenths of a second. The frequency of this band is close to what would be expected if the effective value of d, determined in this case by the aperture and the overhanging rim, is a few millimetres. Some higher bands, particularly those at about 2.2 and 2.6 kHz, are presumably resonances of the lid itself, and their frequency does not change with time. Sonograms of the sound from a spinning plane disc show the same sort of time structure for disc revolution, but lack the low formant band, and all the frequency bands are constant with time.

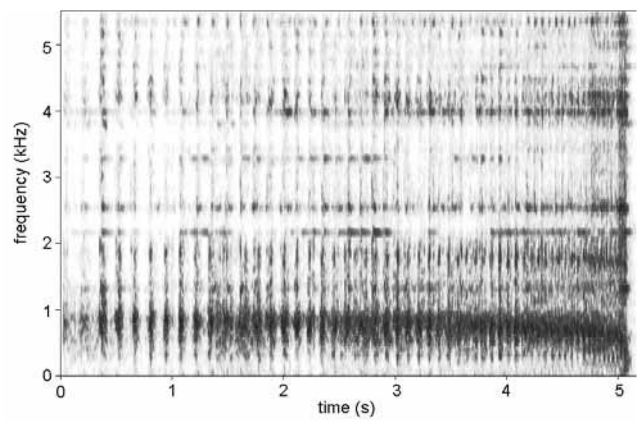

Figure 4: Time-resolved spectral analysis of the sound from a domed metal lid rotating and settling upon a smooth hard plane.

6. CONCLUSIONS

This small study cannot claim any fundamental importance, but was an interesting diversion. One striking thing to emerge was the uniformity of behaviour of the settling rate for bouncing balls, rocking beams, and spinning discs, all leading to a formal infinity in the oscillation rate after a finite time. Because these objects are all things that we encounter from time to time in ordinary life, it is interesting to have some insight into their behaviour.

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